Lior Silberman's Math 223: Problem Set 8 (due 14/3/2022)

Practice problems

Section 4.1, Problems 1-8.
Section 4.2, Problems 1-23 (don't do all of them!)

The determinant of the transpose

1. For a matrix $A \in M_{n,m}(\mathbb{R})$ the *transpose* of A is the matrix $A^t \in M_{m,n}(\mathbb{R})$ such that $(A^t)_{ij} = A_{ji}$.
   (a) Show that the map $A \mapsto A^t$ is linear map and that $(A^t)^t = A$.
   (b) Let A, B be matrices for which the product AB makes sense. Then the product B^tA^t makes sense and $(AB)^t = B^tA^t$.

2. (Elementary matrices) In class we showed that if A is triangular then $\det A = \prod_{i=1}^n a_{ii}$.
   (a) Use this to compute the determinant of the elementary matrices $I_n + cE^{ij}$ and $\mathrm{diag}(a_1, \ldots, a_n)$ (the diagonal matrix with these values on the diagonal).
   (b) Show that if E is an elementary matrix or in row echelon form then $\det(A^t) = \det A$.

*3. Recall the structure theorem of Gaussian elimintation: every $A \in M_n(\mathbb{R})$ can be written in the form $A = E_r \cdot \cdots \cdot E_2 \cdot E_1 \cdot B$ where E_i are elementary and B is in row echelon form. Show that $\det A^t = \det A$ (hint: induction over r).

Some explicit determinants

4. (Vandermonde I) Calculate the following determinants using the definition $V_2(x_1, x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}$, $V_3(x_1, x_2, x_3) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$. Write your answer as a product of linear factors (in other words, factor the polynomials completely).

5. (Tridiagonal I) Calculate the determinants $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$, $\begin{vmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{vmatrix}$, $\begin{vmatrix} a & b & 0 & 0 \\ b & a & b & 0 \\ 0 & b & a & b \\ 0 & 0 & b & a \end{vmatrix}$.

PRAC Can you guess a formula for $V_n(x_1, \ldots, x_n)$, the determinat of the matrix A such that $A_{ij} = x_i^{j-1}$? We will compute the $n \times n$ determinants generalizing 5,6 in the next problem set.

Supplement 1: The Fibbonacci sequence, again

Recall our notation $\mathbb{R}^\infty = \mathbb{R}^\mathbb{N}$ for the space of sequences, and let $L, R \in \text{End}\left(\mathbb{R}^\infty\right)$ be the "shift left" and "shift right" maps:

$$\begin{aligned} L(a_0, a_1, a_2, \ldots) &= (a_1, a_2, \ldots) \\ R(a_0, a_1, a_2, \ldots) &= (0, a_0, a_1, a_2, \ldots) \end{aligned}$$

that is,

$$\begin{aligned} (L\underline{a})_n &= a_{n+1} \\ (R\underline{a})_n &= \begin{cases} 0 & n = 0 \\ a_{n-1} & n \geq 1 \end{cases}. \end{aligned}$$

A. (Basics)
  (a) Find the kernel and image of L, concluding that it is surjective but not injective.
  (b) Find the kernel and image of R, concluding that it is injective but not surjective.
  (c) Show that $LR = \text{Id}$ but that $RL \neq LR$.

B. Let F_n be the sequence defined by $F_0 = a$, $F_1 = b$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.
  (a) Show that $\left(L^2 - L - 1\right)\underline{F} = \underline{0}$.
  (b) Show that the map $\Phi\colon \text{Ker}\left(L^2 - L - 1\right) \to \mathbb{R}^2$ given by $\Phi(\underline{F}) = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$ is an isomorphism of vector spaces.

**C. Show that the set $\left\{R^k L^l \mid k, l \geq 1\right\} \subset \text{End}(\mathbb{R}^\infty)$ is linearly independent.

Supplement 2: Complex numbers

D. Let $\mathbb{C} = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}\right\} \subset M_2(\mathbb{R})$. We will denote elements of $\mathbb{C}$ by lower-case letters like z, w.
  (a) Show that $\mathbb{C}$ is a subspace of $M_2(\mathbb{R})$. Conclude, in particular, that addition in $\mathbb{C}$ satisfies all the usual axioms.
  (b) Show that $\mathbb{C}$ is closed under multiplication of matrices, that $I_2 \in \mathbb{C}$ and that $zw = wz$ for any $z, w \in \mathbb{C}$. It follows that multiplication in $\mathbb{C}$ is associative, commutative, has an identity, and is distributive over addition.
  (c) Use PS5 problem 3 to show that every non-zero $z \in \mathbb{C}$ is invertible and derive a formula for the inverse.
  DEF A set equipped with an addition and a multiplication operations which are commutative, associative, and have neutral elements, satisfying the distributive law and such tha every elemenent has an additive inverse, and every non-zero element has a multiplicative inverse, is called a *field*.
  RMK The field $\mathbb{C}$ constructed above contains a copy of $\mathbb{R}$ – indeed by PS7 problem 3 (practice part) the identification $a \leftrightarrow \begin{pmatrix} a & \\ & a \end{pmatrix}$ respects addition and multiplication of real numbers; we do this from now on. [In fact, we already agreed to identify the number a with the linear map of multiplication by a].

(d) Let $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}$. Show that $i^2 = -1$ (note that $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) and that that every elememt of $\mathbb{C}$ can be uniquely written in the form $a + bi$ for some $a, b \in \mathbb{R}$ (hint: your answer should use the word "basis")

DEF From now on if asked to calculate a complex number write it in the form $a + bi$. Do NOT use the cumbersome specific realization of parts (a)-(d).

RMK Really try to forget the specific construction of parts (a)-(d) and only work in terms of the basis $\{1, i\}$. In particular, note that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$ – you showed this for (b), but it also follows from the applying the distributive law and other laws of arithmetic and at some point using $i^2 = -1$.

(e) Calculate $(1 + 2i) + (3 + 7i)$, $(1 + 2i) \cdot (3 + 7i)$, $\frac{7+3i}{1+2i}$ (hint: division means multiplication by the inverse!)

EXAMPLE $(5-2i)\cdot(1+i) = 5\cdot(1+i)+(-2i)(1+i) = 5+5i-2i-2i\cdot i = 5+3i-2\cdot(-1) = 7+3i$.

E. (Inverting complex numbers using the norm)

DEF The *complex conjugate* of $z \in \mathbb{C}$ is the number $\bar{z}$ represented by the matrix z^t.

(a) Use problem 3 to show $\overline{z+w} = \bar{z} + \overline{w}$ and $\overline{zw} = \overline{zw}$. Also check that $\overline{a+bi} = a - bi$ and use this to give an alternate proof of the claims.

(b) Show that $z\bar{z}$ is a non-negative real for all $z \in \mathbb{C}$ (again we identify $a \in \mathbb{R}$ with the matrix aI_2), and that $z\bar{z} = 0$ iff $z = 0$. Conclude $z \neq 0$ then $z \cdot \frac{\bar{z}}{z\bar{z}} = 1$, a variant of the proof of A(c).

DEF The *norm* of $z\bar{z}$ is defined to be $|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}}$.

(c) Show that $|zw| = |z|\,|w|$. (Hint: this is easy using part (a) of this problem).

(d) Show that $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$.

F. (Linear algebra over the complex numbers)

DEF A *complex vector space* is a triple $(V, +, \cdot)$ satisfying the usual axioms except that multiplication is by complex rather than real numbers.

DEF $\mathbb{C}^X$ is the space of C-valued functions on the set X. This is a complex vector space under pointwise operations (review the definition of $\mathbb{R}^X$). In particular, $\mathbb{C}^n$ is the space of n-tuples.

FACT Everything we proved about real vector spaces is true for complex vector spaces. For example, the standard basis $\{\underline{e}_k\}_{k=1}^n \subset \mathbb{C}^n$ is still a basis. We use $\dim_{\mathbb{C}} V$ to denote the dimension of a complex vector space, and when needed $\dim_{\mathbb{R}} V$ to denote the dimension of a real vector space.

(a) In the vector space $\mathbb{C}^2$ calculate $(1+2i) \cdot \begin{pmatrix} i \\ 3-7i \end{pmatrix}$. Show that $\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$ form a basis for $\mathbb{C}^2$.

(b) Show that $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right\} \subset \mathbb{C}^2$ are linearly independent *over* $\mathbb{R}$ [that is: if a linear combination with real coefficients is zero, then the coefficients are zero].

RMK Since $\begin{pmatrix} a+bi \\ c+di \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} i \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} + d\begin{pmatrix} 0 \\ i \end{pmatrix}$ this set is also spanning,

(c) Solve the following system of linear equations over C:

$$\begin{cases} 5x + iy + (1+i)z & = 1 \\ 2y + iz & = 2 \\ -ix + (3-i)y & = i \end{cases}$$

Supplement 3: computational complexity

G. (Inefficiency of minor expansion) Suppose that the "minor expansion along first row" algorithm for evaluating determinants requires T_n multiplications to evaluate an $n \times n$ determinant.
   (a) Show that $T_1 = 0$ and that $T_{n+1} = (n+1)(T_n + 1)$.
   (b) Show that for $n \geq 2$ one has $T_n = n!\left(\sum_{j=2}^{n} \frac{1}{j!}\right)$
   (c) Conclude that $\frac{1}{2}n! \leq T_n \leq (e-2) \cdot n!$ for all $n \geq 2$.