Lior Silberman's Math 223: Problem Set 8 (due 14/3/2022)

Practice problems

Section 4.1, Problems 1-8.
Section 4.2, Problems 1-23 (don’t do all of them!)

The determinant of the transpose

1. For a matrix $A \in M_{n,m}(\mathbb{R})$ the transpose of $A$ is the matrix $A^t \in M_{m,n}(\mathbb{R})$ such that $(A^t)_{ij} = A_{ji}$.
   (a) Show that the map $A \mapsto A^t$ is linear map and that $(A^t)^t = A$.
   (b) Let $A, B$ be matrices for which the product $AB$ makes sense. Then the product $B^tA^t$ makes sense and $(AB)^t = B^tA^t$.

2. (Elementary matrices) In class we showed that if $A$ is triangular then $\det A = \prod_{i=1}^{n} a_{ii}$.
   (a) Use this to compute the determinant of the elementary matrices $I_n + cE_{ij}$ and $\text{diag}(a_1, \ldots, a_n)$ (the diagonal matrix with these values on the diagonal).
   (b) Show that if $E$ is an elementary matrix or in row echelon form then $\det(A^t) = \det A$.

3. Recall the structure theorem of Gaussian elimination: every $A \in M_n(\mathbb{R})$ can be written in the form $A = E_r \cdots E_2 \cdot E_1 \cdot B$ where $E_i$ are elementary and $B$ is in row echelon form. Show that $\det A^t = \det A$ (hint: induction over $r$).

Some explicit determinants

4. (Vandermonde I) Calculate the following determinants using the definition $V_2(x_1, x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}$,
   $V_3(x_1, x_2, x_3) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$. Write your answer as a product of linear factors (in other words, factor the polynomials completely).

5. (Tridiagonal I) Calculate the determinants $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$, $\begin{vmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{vmatrix}$, $\begin{vmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & b & a \end{vmatrix}$.

PRAC Can you guess a formula for $V_n(x_1, \ldots, x_n)$, the determinant of the matrix $A$ such that $A_{ij} = x_i^{j-1}$?

We will compute the $n \times n$ determinants generalizing 5,6 in the next problem set.
Supplement 1: The Fibonacci sequence, again

Recall our notation \( \mathbb{R}^\infty = \mathbb{R}^\mathbb{N} \) for the space of sequences, and let \( L, R \in \text{End}(\mathbb{R}^\infty) \) be the “shift left” and “shift right” maps:

\[
L(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots) \\
R(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)
\]

that is,

\[
(La)_n = a_{n+1} \\
(Ra)_n = \begin{cases} 
0 & n = 0 \\
a_{n-1} & n \geq 1
\end{cases}
\]

A. (Basics)

(a) Find the kernel and image of \( L \), concluding that it is surjective but not injective.

(b) Find the kernel and image of \( R \), concluding that it is injective but not surjective.

(c) Show that \( LR = \text{Id} \) but that \( RL \neq LR \).

B. Let \( F_n \) be the sequence defined by \( F_0 = a, F_1 = b \) and \( F_{n+2} = F_{n+1} + F_n \) for all \( n \geq 0 \).

(a) Show that \( (L^2 - L - 1) F = 0 \).

(b) Show that the map \( \Phi : \text{Ker}(L^2 - L - 1) \to \mathbb{R}^2 \) given by \( \Phi(F) = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} \) is an isomorphism of vector spaces.

**C. Show that the set \( \{ R^k L^l \mid k, l \geq 1 \} \subset \text{End}(\mathbb{R}^\infty) \) is linearly independent.

Supplement 2: Complex numbers

D. Let \( \mathbb{C} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \subset M_2(\mathbb{R}) \). We will denote elements of \( \mathbb{C} \) by lower-case letters like \( z, w \).

(a) Show that \( \mathbb{C} \) is a subspace of \( M_2(\mathbb{R}) \). Conclude, in particular, that addition in \( \mathbb{C} \) satisfies all the usual axioms.

(b) Show that \( \mathbb{C} \) is closed under multiplication of matrices, that \( I_2 \in \mathbb{C} \) and that \( zw = wz \) for any \( z, w \in \mathbb{C} \). It follows that multiplication in \( \mathbb{C} \) is associative, commutative, has an identity, and is distributive over addition.

(c) Use PS5 problem 3 to show that every non-zero \( z \in \mathbb{C} \) is invertible and derive a formula for the inverse.

DEF A set equipped with an addition and a multiplication operations which are commutative, associative, and have neutral elements, satisfying the distributive law and such that every element has an additive inverse, and every non-zero element has a multiplicative inverse, is called a field.

RMK The field \( \mathbb{C} \) constructed above contains a copy of \( \mathbb{R} \) — indeed by PS7 problem 3 (practice part) the identification \( a \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \) respects addition and multiplication of real numbers; we do this from now on. [In fact, we already agreed to identify the number \( a \) with the linear map of multiplication by \( a \).]
(d) Let $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}$. Show that $i^2 = -1$ (note that $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) and that every element of $\mathbb{C}$ can be uniquely written in the form $a + bi$ for some $a, b \in \mathbb{R}$ (hint: your answer should use the word “basis”).

**DEF** From now on if asked to calculate a complex number write it in the form $a + bi$. Do NOT use the cumbersome specific realization of parts (a)-(d).

**RMK** Really try to forget the specific construction of parts (a)-(d) and only work in terms of the basis $\{1, i\}$. In particular, note that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$ — you showed this for (b), but it also follows from the applying the distributive law and other laws of arithmetic and at some point using $i^2 = -1$.

(e) Calculate $(1 + 2i) + (3 + 7i)$, $(1 + 2i) \cdot (3 + 7i)$, $\frac{7 + 3i}{1 + 2i}$ (hint: division means multiplication by the inverse!)

**EXAMPLE** $(5 - 2i) \cdot (1 + i) = 5 \cdot (1 + i) + (-2i) (1 + i) = 5 + 5i - 2i - 2i \cdot i = 5 + 3i - 2 \cdot (-1) = 7 + 3i$.

### E. (Inverting complex numbers using the norm)

**DEF** The **complex conjugate** of $z \in \mathbb{C}$ is the number $\bar{z}$ represented by the matrix $z' = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(a) Use problem 3 to show $z + \bar{z} = z + \bar{w}$ and that $(z \cdot \bar{z}) = |z|^2$. Also check that $a + b\bar{i} = a - b\bar{i}$ and use this to give an alternate proof of the claims.

(b) Show that $z\bar{z}$ is a non-negative real for all $z \in \mathbb{C}$ (again we identify $a \in \mathbb{R}$ with the matrix $aI_2$), and that $z\bar{z} = 0$ iff $z = 0$. Conclude $z \neq 0$ then $z \cdot \frac{1}{\bar{z}} = 1$, a variant of the proof of A(c).

**DEF** The **norm** of $z\bar{z}$ is defined to be $|z| \overset{\text{def}}{=} \sqrt{z\bar{z}}$.

(c) Show that $|zw| = |z||w|$. (Hint: this is easy using part (a) of this problem).

(d) Show that $\frac{\bar{z}}{w} = \frac{\bar{w}}{|w|^2}$.

### F. (Linear algebra over the complex numbers)

**DEF** A **complex vector space** is a triple $(V, +, \cdot)$ satisfying the usual axioms except that multiplication is by complex rather than real numbers.

**DEF** $\mathbb{C}^X$ is the space of $\mathbb{C}$-valued functions on the set $X$. This is a complex vector space under pointwise operations (review the definition of $\mathbb{R}^X$). In particular, $\mathbb{C}^n$ is the space of $n$-tuples.

**FACT** Everything we proved about real vector spaces is true for complex vector spaces. For example, the standard basis $\{e_k\}_{k=1}^n \subset \mathbb{C}^n$ is still a basis. We use $\dim_{\mathbb{C}} V$ to denote the dimension of a complex vector space, and when needed $\dim_{\mathbb{R}} V$ to denote the dimension of a real vector space.

(a) In the vector space $\mathbb{C}^2$ calculate $(1 + 2i) \cdot \begin{pmatrix} i \\ 3 - 7i \end{pmatrix}$. Show that $\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$ form a basis for $\mathbb{C}^2$.

(b) Show that $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right\} \subset \mathbb{C}^2$ are linearly independent over $\mathbb{R}$ [that is: if a linear combination with real coefficients is zero, then the coefficients are zero].

**RMK** Since $\begin{pmatrix} a + bi \\ c + di \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} i \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} 0 \\ i \end{pmatrix}$ this set is also spanning.

(c) Solve the following system of linear equations over $\mathbb{C}$:

\[
\begin{align*}
5x + iy + (1 + i)z &= 1 \\
2y + iz &= 2 \\
-ix + (3 - i)y &= i
\end{align*}
\]
Supplement 3: computational complexity

G. (Inefficiency of minor expansion) Suppose that the “minor expansion along first row” algorithm for evaluating determinants requires $T_n$ multiplications to evaluate an $n \times n$ determinant.

(a) Show that $T_1 = 0$ and that $T_{n+1} = (n + 1)(T_n + 1)$.

(b) Show that for $n \geq 2$ one has $T_n = n! \left( \sum_{j=2}^{n} \frac{1}{j!} \right)$.

(c) Conclude that $\frac{1}{2} n! \leq T_n \leq (e - 2) \cdot n!$ for all $n \geq 2$. 