Lior Silberman’s Math 223: Problem Set 10 (due 28/3/2022)

Practice problems

Section 5.1: all problems are suitable
Section 5.2: all problems are suitable

Calculation

M1. Find the characteristic polynomial of the following matrices.

(a) \[
\begin{pmatrix}
5 & 7 \\
-3 & 2
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
\pi & e \\
\sqrt{7} & 0
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
0 & 1 \\
0 & 1 \\
\vdots & \ddots \\
0 & 1
\end{pmatrix}
- \begin{pmatrix}
a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1}
\end{pmatrix}
\]

M2. For each of the following matrices find its spectrum and a basis for each eigenspace.

(a) \[
\begin{pmatrix}
5 & 4 & 2 \\
4 & 5 & 2 \\
2 & 2 & 2
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 2
\end{pmatrix}
\]

Projections

Fix a vector space \( V \).

1. Let \( T, T' \in \text{End}(V) \) be similar. Show that \( p_T(x) = p_{T'}(x) \). (Hint: show that \( x \text{Id} - T, x \text{Id} - T' \) are similar)

2. Let \( T \in \text{End}(V) \).
   (a) Let \( p \in \mathbb{R}[x] \), and let \( v \in V \) be an eigenvector of \( T \) with eigenvalue \( \lambda \). Show that \( v \) is an eigenvector of \( p(T) \) with eigenvalue \( p(\lambda) \).
   (b) Suppose \( p(T) = 0 \). Show that \( p(\lambda) = 0 \) for all eigenvalues \( \lambda \) of \( V \).
   (c) Show that the only eigenvalue of a nilpotent map is 0.

3. Let \( P \in \text{End}(V) \) satisfy \( P^2 = P \). Such maps are called projections.
   (a) Apply problem 2(b) to show that \( \text{Spec}(P) \subset \{0, 1\} \).
   (b) Let \( V_0 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \) and \( V_1 = \text{Span} \left\{ \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \) so that \( \mathbb{R}^3 = V_0 \oplus V_1 \) [no need to check this separately]. Let \( P \) be the projection onto \( V_1 \) along \( V_0 \). Find the matrix of \( P \) with respect to the standard basis of \( \mathbb{R}^3 \).
   
   \text{Hint: By diagonalization } P = S \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} S^{-1} \text{ where } S \text{ is the matrix of eigenvectors.}
The Quantum Harmonic Oscillator, I

PRAC In physics a “parity operator” is a map \( R \in \text{End}(V) \) such that \( R^2 = I \) (we use the shorthand \( I = \text{Id}_V \)).

RMK This was problem 4, but it is for practice, not for submission.

(a) Show that \( \pm I \) are (uninteresting) parity operators.

— For parts (b)-(d) fix a parity operator \( R \).

(b) Show that the eigenvalues of \( R \) are in \( \{ \pm 1 \} \); let \( V_\pm \) be the corresponding eigenspaces.

(c) Show that \( I + R \), \( I - R \) are the projections onto \( V_+ \), \( V_- \) along the other subspace, respectively.

\[ \text{Hint: compute } (I + R)^2 \text{ using that } R^2 = I. \]

(d) Conclude that \( V = V_+ \oplus V_- \) and hence that every parity operator is diagonalizable.

(e) Let \( X \) be a set and let \( \tau : X \to X \) be an involution: a map such that \( \tau^2 = \text{id}_X \) (identity permutation).

Let \( R_{\tau} \in \text{End}(\mathbb{R}^X) \) be the linear map \( f \mapsto f \circ \tau \). Show that \( P_{\tau} \) is a parity operator.

(f) Let \( X = \mathbb{R} \), \( \tau(x) = -x \). Explain how (b)-(e) relate to the concepts of odd and even functions.

5. Let \( V = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \right\} \) and for \( n \geq 1 \) let \( V_n = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \lt n \right\} \subset V \). Let \( H \in C^\infty(\mathbb{R}) \) be the operator (“quantum Hamiltonian”) \( H = -D^2 + Mx^2 \). Concretely we have \( Hf = -\frac{d^2f}{dx^2} + x^2f \).

PRAC Show that \( V_n \subset V \) are subspaces of \( C^\infty(\mathbb{R}) \), the space of infinitely differentiable functions.

(a) Show that \( HV \subset V \) and \( HV_n \subset V_n \).

(b) Let \( H_n = H \mid_{V_n} \in \text{End}(V_n) \) be the restriction of \( H \) to \( V_n \). Show that \( H_n \) has an upper-triangular basis with respect to an appropriate basis of \( V_n \) and determine its eigenvalues.

(c) Show that \( H_n \) is diagonalizable.

(d) Show that \( HR = RH \) for the parity operator of 4(f).

(*e) Show that every eigenfunction of \( H_n \) is either even or odd. Which is which?

(f) Show that \( V = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \right\} \) has a basis of eigenfunctions of \( H \), and that each eigenfunction is either even or odd.
Extra credit: the generalized eigenvalue decomposition and the Cayley–Hamilton Theorem

Fix a vector space \( V \) and a linear map \( T \in \text{End}_F(V) \).

A. DEF For a number \( \lambda \) define the generalized \( \lambda \)-eigenspace to be the set of vectors \( v \in V \) killed by some power of \( T - \lambda \) (possibly depending on \( v \)):

\[
\tilde{V}_\lambda = \left\{ v \in V : \exists k : (T - \lambda)^k v = 0 \right\}.
\]

(a) Show that \( \tilde{V}_\lambda \) is a subspace containing \( V_\lambda \).
(b) Show that \( \tilde{V}_\lambda \neq \{0\} \) iff \( V_\lambda \neq \{0\} \) (“every generalized eigenvalue is a regular eigenvalue”).
(c) Show that \( V_\lambda \) and \( \tilde{V}_\lambda \) are \( T \)-invariant: if \( v \in \tilde{V}_\lambda \) then \( T v \in \tilde{V}_\lambda \) as well, and similarly for \( V_\lambda \).
(d) Let \( \mu \neq \lambda \). Show that \( T |_{\tilde{V}_\lambda} - \mu \in \text{End}(\tilde{V}_\lambda) \) is injective (“no other eigenvalues in \( \tilde{V}_\lambda \) except \( \lambda \”). Using a factorization into linear terms conclude that for any polynomial \( p \) if \( p(\lambda) \neq 0 \) then \( p \left( T |_{\tilde{V}_\lambda} \right) \in \text{End}(\tilde{V}_\lambda) \) is injective there.
(e) Show that \( \{ \tilde{V}_\lambda \}_{\lambda \in \text{Spec}(T)} \) are linearly independent.

COR The sum \( \tilde{V} = \bigoplus_{\lambda \in \text{Spec}(T)} \tilde{V}_\lambda \) is direct.

B. Continuing the previous problem, suppose now that \( V \) is finite-dimensional.

(a) Show that \( p_T |_{\tilde{V}_\lambda} (x) = (x - \lambda)^{\dim \tilde{V}_\lambda} \) and that \( \left( T |_{\tilde{V}_\lambda} - \lambda \right)^{\dim \tilde{V}_\lambda} = 0 \tilde{V}_\lambda \).
(b) Let \( m(x) = \prod_{\lambda \in \text{Spec}(T)} (x - \lambda)^{\dim \tilde{V}_\lambda} \). Show that \( m(x) = p_T |_{\tilde{V}_\lambda} (x) \) and that \( m(T |_{\tilde{V}_\lambda}) = 0 \).
(c) Suppose that \( \tilde{V} \neq V \). Show that setting \( \bar{T} (v + \tilde{V}) = T v + \tilde{V} \) gives a well-defined linear map \( \bar{T} \) on the quotient vector space \( W = V / \tilde{V} \).
(d) Let \( \mu \) be a root of \( p_T(x) \), and let \( W_\mu \subset W \) be the corresponding eigenspace. Show that \( \prod_{\lambda \in \text{Spec}(T) \setminus \{\mu\}} (\bar{T} - \lambda)^{\dim \tilde{V}_\lambda} \) is an invertible map there. Conclude that if \( v + \tilde{V} \in W_\mu \) with \( v \notin \tilde{V} \) then \( u = \prod_{\lambda \in \text{Spec}(T) \setminus \{\mu\}} (T - \lambda)^{\dim \tilde{V}_\lambda} v \notin \tilde{V} \) but \( u + \tilde{V} \in W_\mu \).
(e) Suppose \( \mu \) is not an eigenvalue of \( T \). Show that \( (T - \mu) u = 0 \), a contradiction to \( u \notin \tilde{V} \).
    
    Hint: In this case the polynomial in the definition of \( u \) is exactly \( m(T) \).
(f) Suppose \( \mu \) is an eigenvalue of \( T \). Show that \( (T - \mu)^{1 + \dim V_\mu} u = 0 \) showing that \( u \in \tilde{V}_\mu \subset \tilde{V} \), a contradiction.

C. It follows that \( V = \tilde{V} \) so that\( T |_{\tilde{V}} = T \). Problem B(b) now gives two corollaries:

(a) The algebraic multiplicity of \( \lambda \in \text{Spec}(T) \) is equal to \( \dim \tilde{V}_\lambda \) (and since \( V_\lambda \subset \tilde{V}_\lambda \) we get a new proof that the algebraic multiplicity is at least the geometric multiplicity).
(b) (Cayley–Hamilton Theorem) \( p_T(T) = 0 \).