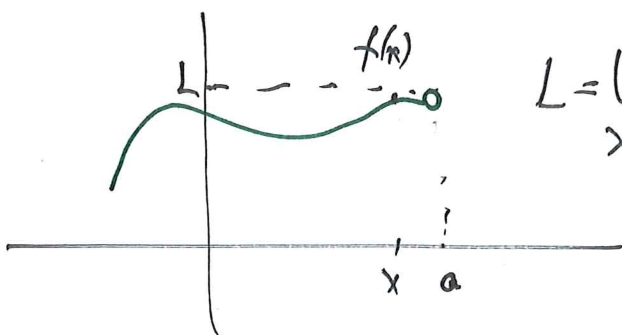
3. The Derivative (29/9/2022)

Goals.

(1) The derivative at a point
(2) Tangent lines & linear approximations
(3) The derivative as a function

Last Time. Limits

Graphically: $\lim_{x\to a} f(x)$ is the value $f(x)$ "tends to" as $x \to a$

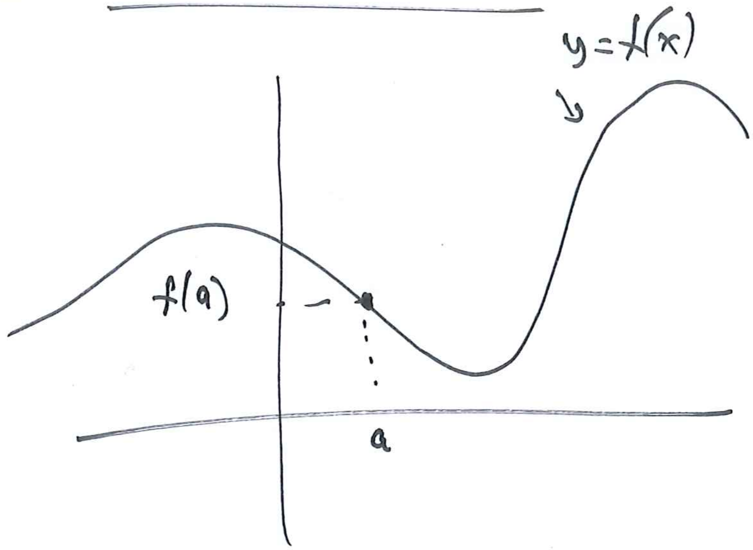


$L = \lim_{x\to a} f(x)$

Algebraically: If f given by formula, defined at a, then $\lim_{x\to a} f(x) = f(a)$ ("just plug in")

Philosophically: limit is where $f(x)$ is going
asymptotics is about how $f(x)$ gets there

① The Derivative

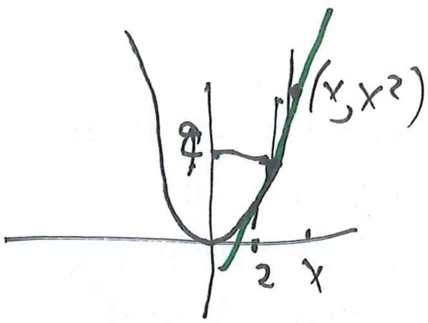


- f <u>continuous</u> at a ⟹ if x close to a, f(x) close to f(a)

Q: How close is "close"?
if we "wiggle" input to x (close to a)
then f(x) will "wiggle" around f(a)

WS 1

notes:
(1) proportional thinking
(2) plugging anything into functions
(3) compare quantities using subtraction



Math 100C – WORKSHEET 3
THE DERIVATIVE

1. Three views of the derivative

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2, 4)$ is a point on the graph of $y = f(x)$.

(a) Let (x, x^2) be another point on the graph, close to $(2, 4)$. What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?

slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{x^2 - 4}{x - 2} = x + 2 \xrightarrow[x \to 2]{} 4$

$\Delta y \approx (\text{slope}) \cdot \Delta x$

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2 + h)$ as $h \to 0$? What aobut $f(2 + h) - f(2)$?

$f(2+h) = (2+h)^2 \sim 2^2 = 4$ as $h \to 0$

$f(2+h) - f(2) = (2+h)^2 - 4 = 4h + h^2 \sim 4h$ (as $h \to 0$)

$\iff f(2+h) \approx f(2) + 4h$ (wiggle by h, $\Delta y \approx 4h$)

"linear approximation to f near $x=2$"

(c) What is $\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$?

$\dfrac{(2+h)^2 - 2^2}{h} = \dfrac{4h + h^2}{h} = 4 + h \xrightarrow[h \to 0]{} 4$

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2, 4)$?

Date: 29/9/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$y = \underbrace{4}_{\text{slope}}(x - 2) + \underbrace{4}_{\text{point } (2,4)}$ or $y - 4 = 4(x - 2)$

or $y = 4x - 4$

Def: The derivative of f at $x=a$ is the limit

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

view (a)   if $h = x-a$ then $x = a+h$ view (c)

$$\Longleftrightarrow \begin{cases} f(a+h) \approx f(a) + f'(a)\cdot h \\ f(a+h) - f(a) \approx f'(a)\cdot h \\ f(x) \approx f(a) + f'(a)(x-a) \end{cases}$$

← view (b)

Moral: A "nice function" is approximately linear (on small scales) ⇒ has a slope.

Observation: if $f'(a) > 0$ then f is increasing near a
if $f'(a) < 0$ " " " decreasing near a

(Correction to f is $f'(a)\cdot h$, has same sign as h if $f'(a) > 0$, opposite if $f'(a) < 0$)

Calculating derivatives

1 (c) A firm's profit function

$$P(x) = 10x(7-x) - 3x - 5$$

(if x units are produced)

$$P(2+h) = 10(2+h)(7-(2+h)) - 3(2+h) - 5$$
$$= 10(2+h)(5-h) - 11 - 3h$$
$$= 100 + 30h - 11 - 3h - 10h^2$$
$$= 89 + 27h - 10h^2 \approx 89 + 27h$$

to linear order in h

$P(2) = 20(5) - 6 - 5 = 89$

$\Rightarrow$ increase production ($P'(2) = 27$)

Econ lingo: the **marginal profit** is $\dfrac{27}{\text{marginal unit}}$

Easier: $P(x) = 70x - 10x^2 - 3x - 5$

$P'(x) = 70 - 20x - 3$, $P'(2) = 27$

2. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a+h) \approx f(a) + f'(a)h$

(3) Find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

$$\lim_{x\to 3} \frac{x^2-3^2}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} x+3 = 6$$

$$(3+h)^2 = 9 + 6h + h^2 \approx 9 + 6h \quad \text{so } f'(3) = 6$$

to 1st order in h

$$(3+h)^2 - 9 = 6h + h^2 \sim 6h \quad \text{as } h\to 0 \text{ so } f'(3) = 6$$

(b) $f(x) = \frac{1}{x}$, any a.

$$\frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \frac{\frac{a}{a(a+h)} - \frac{a+h}{a(a+h)}}{h} = \frac{\frac{-h}{a(a+h)}}{h} = -\frac{1}{a(a+h)} \xrightarrow{h\to 0} -\frac{1}{a^2}$$

Or:

$$\frac{1}{a+h} - \frac{1}{a} = -\frac{h}{a(a+h)} \sim -\frac{h}{a^2} \quad \text{as } h\to 0$$

$$\text{so } f'(a) = -\frac{1}{a^2}$$

13) (a) Express the limits as derivatives:

$\lim_{h \to 0} \frac{\cos(5+h) - \cos 5}{h}$ this is the derivative of $f(x) = \cos x$ at $x = 5$

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0} = \left[\frac{d}{dx} \sin x\right]_{x=0}$

$\sin 0 = 0$

(notation: the derivative of f is denoted $f'(x)$, $D_x f$, Df, $\frac{df}{dx}$, $\frac{d}{dx} f$, ..)

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recap: differentiation $\leftrightarrow$ local behavior of $f(x)$ when we change x.

has definition (as limit)

connected to tangent lines to graphs

(approximate) linear behaviour of f)

3. The tangent line

(6) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

$f(x) = x^{\frac{1}{2}}$ so $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, so $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

so the line is

$$y = \frac{1}{4}(x-4) + 2$$

(line of slope $\frac{1}{4}$ through $(4,2)$)

(7) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

The point of tangency is $(1, 4\cdot 1 + 2) = (1, 6)$

if $f(x) = x^3 + x^2 - x$, $f(1) = 1$ (line doesn't intersect graph of f at $x = 1$)

same $\left[x^3 + x + 2\right]_{x=1} = 4$

for $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$

compute slopes at $x = 1$, check if get 4.

4. Linear approximation

Definition. $f(a+h) \approx f(a) + f'(a)h$

(10) Estimate

(a) $\sqrt{1.2}$

Let $f(x) = \sqrt{x}$, work near $a = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(1) = \frac{1}{2}$

so $\sqrt{1.2} \approx \sqrt{1} + \frac{1}{2}\cdot(1.2-1) = 1.1$

to 1st order

(b) (Final, 2015) $\sqrt{8}$

Now work about $a = 9$

$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

so $f(8) \approx \sqrt{9} + \frac{1}{6}(-1) = 3 - \frac{1}{6} = 2\frac{5}{6}$