8. Differential Equations (3/11/2022)

Goals.

(1) Taylor expansion
  (a) Review
  (b) Combining expansion

(2) Differential equations
  (a) What is a differential equation
  (b) Plugging ansatze into equations

Last Time.

Optimization: ① Given some set-up we can find a mathematical description;

② We can find max/min of a function on an interval, by evaluating f on critical, singular, end points of interval (if f undef at endpoints need extra arguments)

critical: $f'(x_0)=0$ singular: $f'(x_0)$ DNE

Review: Can approximate $f(x)$ near $x=a$ by polynomial

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Math 100C – WORKSHEET 8
DIFFERENTIAL EQUATIONS

1. Manipulating Taylor expansions

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

In addition we have the following expansions about $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ; \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

(1) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

$(\sin x)' = \cos x$, $(\sin x)^{(2)} = -\sin x$; $(\sin x)^{(3)} = -\cos x$;

so $\sin 0 = 0$, $\cos 0 = 1$, 3rd order expansion of $\sin x$ about 0 is

$$0 + \frac{1}{1!}\cdot x + \frac{0}{2!}\cdot x^2 - \frac{1}{6}x^3 = x - \frac{1}{6}x^3$$

so $\sin\frac{1}{100} \approx \frac{1}{100} - \frac{1}{6\cdot 10^6}$

Since $\sin x \approx x - \frac{x^3}{6}$ to 3rd order about 0,
$x+1 = 1+x$ " " " " "

Date: 3/11/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Then $(1+x)\sin x \approx (1+x)\left(x - \frac{x^3}{6}\right) = x + x^2 - \frac{x^3}{6} - \frac{x^4}{6}$

$\approx x + x^2 - \frac{x^3}{6}$ to 3rd order

Idea: If f made of pieces, approximate each piece, then combine the approximations (here by multiplication)

(2) If f is a sum, approximate each piece and add them together.

Observation: Expansion of x about 4 is:

$$x = (x-4)+4 = 4+(x-4)$$

$$1+x = 1+(4+(x-4)) = 5+(x-4)$$

$$x^2 = (4+(x-4))^2 = 16+8(x-4)+(x-4)^2$$

$\vdots$

(can always find expansion by differentiation, but often can also do some algebra instead)

Idea of change of variable: switch from x to $x-4$.

(3) Expand $\frac{e^{x^2}}{1+x}$ to second order about $x = 1$.

$$\frac{e^{x^2}}{1+x} = e^{x^2} \cdot \frac{1}{1+x}$$

To approximate e^{x^2}: know $e^u \approx 1 + u + \frac{u^2}{2}$

$$x^2 = (1+(x-1))^2 = 1 + 2(x-1) + (x-1)^2$$

small as $x \to 1$

So $e^{x^2} = e^{1+2(x-1)+(x-1)^2} = e \cdot e^{2(x-1)+(x-1)^2}$ (where $u = 2(x-1)+(x-1)^2$)

so to 2nd order $e^{x^2} = e \cdot e^{2(x-1)+(x-1)^2} \approx e\left(1 + (2(x-1)+(x-1)^2) + \frac{(2(x-1)+(x-1)^2)^2}{2}\right)$

($u^2/2!$)

$$\approx e\left(1 + 2(x-1) + (x-1)^2 + 2(x-1)^2 + \text{higher order terms}\right)$$

$$= e\left(1 + 2(x-1) + 3(x-1)^2\right)$$

Similarly,

$$\frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2}\,\frac{1}{1+\frac{x-1}{2}} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{x-1}{2}\right)}$$

$$\approx \frac{1}{2}\left(1 + \left(-\frac{x-1}{2}\right) + \left(-\frac{x-1}{2}\right)^2 + \dots\right)$$

So
$$\frac{e^{x^2}}{1+x} \approx e\left(1 + 2(x-1) + 3(x-1)^2\right)\frac{1}{2}\left(1 - \frac{x-1}{2} + \frac{1}{4}(x-1)^2\right)$$

Example; Expand $\sqrt{\cos x}$ about $x=0$

Solution; To 4th order (say) have

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

so $\sqrt{\cos x} \approx \sqrt{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} \approx \sqrt{y}$, $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

close to 1 if $x \to 0$

if $f(y) = \sqrt{y}$ then $f'(y) = \frac{1}{2\sqrt{y}}$, $f''(y) = -\frac{1}{4y^{3/2}}$ so

$f(1) = 1$, $f'(1) = \frac{1}{2}$ $f''(1) = -\frac{1}{4}$

so $f(y) \approx 1 + \frac{1}{2}(y-1) - \frac{1}{4}(y-1)^2$

so $\sqrt{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} \approx 1 + \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{4}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2$

$\approx 1 - \frac{1}{4}x^2 + \frac{1}{48}x^4 - \frac{1}{16}x^4 +$ higher order

so $\boxed{\sqrt{\cos x} \approx 1 - \frac{1}{4}x^2 - \frac{1}{24}x^4}$ to 4th order

Differential equations

A differential equation is an equation where:

(1) the unknown is a function
(the equation is an equality of functions)

(2) the equation involves values of the function and its derivatives

Example: $y' = 0$ or $\frac{dy}{dx} = 0$

the functions $y \equiv 0$, $y \equiv 1$, $y \equiv -\pi$ all work, ← particular solutions

in general $y = C$ (C constant) works

↑ "the general solution"

Example: $F = ma \iff \frac{d^2 x(t)}{dt} = \frac{F(x(t))}{m}$

2. Differential equations

(6) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \quad ; \quad y^2 = 3y$$

$3^2 = 9 \neq 4$ ✗ $\quad 3^2 = 9 = 3 \cdot 3$ ✓

$2^2 = 4$ ✓ $\quad 2^2 = 4 \neq 3 \cdot 2$ ✗

all solutions: ± 2 $\quad$ all solutions: $0, 3$

(7) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \quad ; \quad \left(\frac{dy}{dx}\right)^2 = 4y$$

$\frac{d(x^2)}{dx} = 2x \neq x^2$ ✗ $\quad \left(\frac{d(x^2)}{dx}\right)^2 = 4x^2 = 4 \cdot (x^2)$ ✓

$\frac{d(e^x)}{dx} = e^x$ ✓ $\quad \left(\frac{d(e^x)}{dx}\right) = (e^x)^2 = e^{2x} \neq 4 \cdot e^x$ ✗

(8) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

$$\frac{d(t^2)}{dt} + t^2 - 1 = 2t + t^2 - 1 \neq t^2$$

by $$\frac{d(t^2+2t+1)}{dt} + t^2 - 1 = (2t+2) + (t^2 - 1) = t^2 + 2t + 1 \quad \checkmark$$

so $z(t) = t^2 + 2t + 1$ is a solution but $z(t) = t^2$ isn't.

General solution: $z(t) = t^2 + 2t + 1 + Ce^t$

C arbitrary.

(Exs if z, w two solutions then $\frac{d(w-z)}{dt} = (w-z)$

Our bank balance earns 4% interest, compounded continuously, i.e. satisfies $y' = (1.04) \cdot y$

(1) what is the general solution? it is $y = Ce^{1.04t}$

(know this DE)

(2) the solution with $y(0) = \$100$ is $y(t) = 100e^{1.04t}$ dollars

idea: an initial value constraint (about $y(0)$, $y(1)$, ..) gives algebraic equation for c.

3. Solutions by massaging and ansatze

(11) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

$$\frac{d^2(\sin(wt))}{dt^2} = -w^2 \sin(wt)$$

so

$$\frac{d^2(\sin(wt))}{dt^2} + 4\sin(wt) = (4 - w^2)\sin(wt)$$

If $w = \pm 2$ get a solution (also if $w=0$ get $y \equiv 0$)

otherwise we don't.