Math 100C – WORKSHEET 6
CURVE SKETCHING; TAYLOR EXPANSION

1. Curve sketching

Let $f(x) = \frac{x^3+2}{x^2+1}$; and that $f''(x) = -2\frac{x^3-6x^2-3x+2}{(x^2+1)^3}$

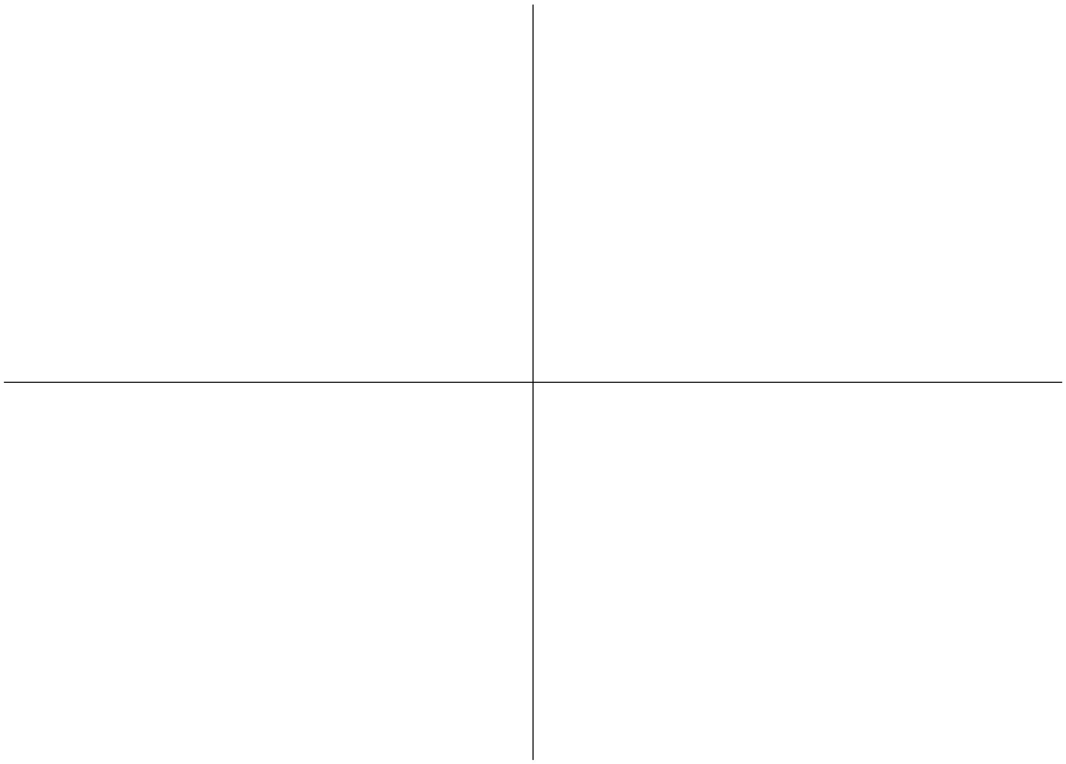
(1) Zeroeth derivative questions
   (a) Where is f defined?
   (b) List the vertical asymptotes of f, if any?
   (c) What are the asymptotic behaviours of f at $\pm\infty$?
   (d) Where does f meet the axes?
(2) It is a fact that $f'(x) = \frac{x(x-1)(x^2+x+4)}{(x^2+1)^2}$ (in an exam you might be asked to differentiate the function yourself)enumerate
(3) Where is f differentiable?
(4) Where does $f'(x) = 0$? Where it is positive? Negative?
(5) Where are the local extrema of f? What are the values at those points?

It is a fact that $f''(x) = -2\frac{x^3-6x^2-3x+2}{(x^2+1)^3}$.

(1) Where is f'' positive/negative? Where does it vanish? Say as much as you can.
(2) Where is f concave up/down? Where are its inflection points?

Date: 20/10/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Draw a sketch of the graph of f, incorporating all the features you have identified in questions 1-3.



- Extra credit: Find the constant b so that $f(x) \approx x + b$ as $x \to \infty$ (in the sense that $f(x) - x - b \to 0$). We call this line a *slant asymptote* for f.

2. Taylor expansion

(5) (Review) Use linear approximations to estimate:

(a) $\log \frac{4}{3}$ and $\log \frac{2}{3}$. Combine the two for an estimate of $\log 2$.

(b) $\sin 0.1$ and $\cos 0.1$.

(6) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(7) Do the same with $f(x) = \log x$ about $x = 1$.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n$$

(8) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

(9) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

(10) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

(11) In labour economics, the *CES production function* is the functional form $Q(K, E) = \left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta}$. Here K is capital, E is employment, and $\delta < 1$ measures the degree of substitution between labour and capital. Find the linear and quadratic expansions of Q in the variable E about the point $(K_0, E_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$ if $\alpha = \frac{1}{2}$.

3. New expansions from old

(12) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

(13) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(14) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

(15) Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.