Math 100C – SOLUTIONS TO WORKSHEET 11
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

(1) How many critical points does $f(x,y) = x^2 - x^4 + y^2$ have?

Solution: $\frac{\partial f}{\partial x}(x,y) = 2x - 4x^3 = 2x(1-2x^2)$ while $\frac{\partial f}{\partial y} = 2y$. Thus $\frac{\partial f}{\partial y} = 0$ only when $y = 0$ while $\frac{\partial f}{\partial x} = 0$ when $x \in \left\{0, \pm\frac{1}{\sqrt{2}}\right\}$. Thus there are three critical points: $(0,0)\,, \left(0, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}\right)$.

(2) Find the critical points of $f(x,y) = x^2 - x^4 + xy + y^2$.

Solution: Now $\frac{\partial f}{\partial x}(x,y) = 2x - 4x^3 + y$ while $\frac{\partial f}{\partial y} = x + 2y$. At a critical point we have $\frac{\partial f}{\partial y} = 0$ so $y = -\frac{1}{2}x$ and also $\frac{\partial f}{\partial x}(x,y) = 0$ so $2x - 4x^3 + y = 0$. Substituting $y = -\frac{1}{2}x$ we get $\frac{3}{2}x - 4x^3 = 0$ or $-4x\left(x^2 - \frac{3}{8}\right) = 0$ so we have a critical point when $x \in \left\{0, \pm\sqrt{\frac{3}{8}} = \pm\frac{1}{2}\sqrt{\frac{3}{2}}\right\}$ and hence at the points $\left\{(0,0)\,, \left(\frac{1}{2}\sqrt{\frac{3}{2}}, -\frac{1}{4}\sqrt{\frac{3}{2}}\right), \left(-\frac{1}{2}\sqrt{\frac{3}{2}}, \frac{1}{4}\sqrt{\frac{3}{2}}\right)\right\}$.

(3) (MATH 105 Final, 2013) Find the critical points of $f(x,y) = xye^{-2x-y}$.

Solution: $\frac{\partial f}{\partial x}(x,y) = ye^{-2x-y} - 2xye^{-2x-y} = y(1-2x)e^{-2x-y}$ while $\frac{\partial f}{\partial y}(x,y) = xe^{-2x-y} - xye^{-2x-y} = x(1-y)e^{-2x-y}$. Since $e^{-2x-y} \neq 0$ everywhere, the critical points are the solutions to the system of equations

$$\begin{cases} y(1-2x) = 0 \\ x(1-y) = 0 \end{cases}.$$

Starting with the second equation we either have $x = 0$ or $y = 1$. In the first case the first equation reads $y = 0$ and we get the critical point $(0,0)$. In the second case the first equation reads $1 - 2x = 0$ and we get the critical point $\left(\frac{1}{2}, 1\right)$.

(4)

(a) Let $f(x,y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

Solution: $\frac{\partial f}{\partial x} = 8x$ and $\frac{\partial f}{\partial y} = 16y$. The only point where both vanish is where $x = y = 0$ where $f(0,0) = 7$. Since $4x^2 + 8y^2 \geq 0$ for all x, y we have $f(x,y) \geq 7$ for all x, y so this point is the global minimum, and in particular a local minimum.

(b) (MATH 105 Final, 2017) Let $f(x,y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

Solution: $\frac{\partial f}{\partial x} = -8x$ and $\frac{\partial f}{\partial y} = 16y$. The only point where both vanish is where $x = y = 0$ where $f(0,0) = -3$. We have a local *maximum* along the x axis (for constant y the parabola $-4x^2 + \left(8y^2 - 3\right)$ is concave down) but a local *minimum* along the y axis (for constant x the parabola $8y^2 - \left(4x^2 + 3\right)$ is concave up), so this is a saddle point.

(5) Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.

Solution: Since $\frac{\partial f}{\partial x} = e^{-x-y}(7 - 7x - 3y - 2y^2)$ and $\frac{\partial f}{\partial y} = e^{-x-y}(3 + 4y - 7x - 3y - 2y^2)$ the critical points are at

$$\begin{cases} 7x + 3y + 2y^2 &= 7 \\ 7x + 3y + 2y^2 &= 3 + 4y\,. \end{cases}$$

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At a solution of this system we must have $3+4y=7$ so $y=1$ and then $7x=7-3y-2y^2$ forces $x=\frac{2}{7}$, so the only critical point is at $\left(\frac{2}{7},1\right)$.

2. Optimization

(6) Find the maximum of $(7x+3y+2y^2)e^{-x-y}$ for $x\geq 0$, $y\geq 0$,

Solution: We start with the boundary. If $y=0$ we have $f(x,0)=7xe^{-x}$, the derivative of which is $7e^{-x}-7xe^{-x}=7(1-x)e^{-x}$ which only vanishes at $x=1$. The maximum is then at $x=1$ where the value is $\frac{7}{e}$. If $x=0$ we get $f(0,y)=(3y+2y^2)e^{-y}$ with derivative $(3+4y-3y-2y^2)e^{-y}=-\left(2y^2-y-3\right)e^{-y}$. This vanishes at $y=\frac{1\pm\sqrt{25}}{4}=\frac{3}{2},-1$, so at $y=\frac{3}{2}$. Since $f(0,0)=0$, $f(0,\frac{3}{2})=9e^{-3/2}>0$ and $f(0,y)$ is negative for large y, the maximum on this boundary is at $9e^{-3/2}$. Finally the function tends to 0 if $x\to\infty$ or $y\to\infty$ (the exponential always wins) so there will be a maximum which, if it occurs at the interior, must occur at a critical point. We already say that the only critical point is at $\left(\frac{2}{7},1\right)$, and evaluation gives $f(\frac{2}{7},1)=7e^{-9/7}<\frac{7}{e}$. The maximum is therefore at the larger of the boundary values. Now

$$\left(\frac{7}{e}/\frac{9}{e^{3/2}}\right)^2=\frac{7^2e}{9^2}>\frac{49\cdot 2}{81}>1$$

so $\frac{7}{e}$ is the largest value, hence the maximum. (With a calculator we could also check that $\frac{7}{e}\approx 2.58$, $\frac{9}{e^{3/2}}\approx 2.01$, and $\frac{7}{e^{9/7}}\approx 1.94$).

(7) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C=3t^2+\sqrt{t}\cdot q$. At price p the company can sell $q=\frac{t-p}{3}$ widgets.

(a) Write an expression for the profit function $f(q,t)$.

Solution: To sell q widgets the price must be $p=t-3q$, so the revenue will be $R=qp=tq-3q^2$ and the profit will be

$$f(q,t)=R-C=tq-3q^2-3t^2+\sqrt{t}\cdot q\,.$$

(b) How many widgets of what quality should the company make to maximize profits?

Solution: We need to maximize

$$f(q,t)=tq-3q^2-3t^2+\sqrt{t}\cdot q\,.$$

Now $\frac{\partial f}{\partial q}=t-6q+\sqrt{t}$ while $\frac{\partial f}{\partial t}=q\left(1+\frac{1}{2\sqrt{t}}\right)-6t$. From the first equation we find that at fixed quality we maximize profits at $q=\frac{t+\sqrt{t}}{6}$. As $t\to\infty$ $q\sim\frac{t}{6}$ so

$$\begin{aligned}f(q,t)&\sim t\cdot\frac{t}{6}-3\left(\frac{t}{6}\right)^2-3t^2+\sqrt{t}\frac{t}{6}\\&\sim-\left(3-\frac{1}{6}\right)t^2\to-\infty\end{aligned}$$

so there is a limit to the qualities at which we will make a profit. Conversely at quality 0 we have $f(q,0)=-3q^2\leq 0$ so we must have some positive quality to make a profit, and the maximum will occur at a critical point. Plugging $q=\frac{t+\sqrt{t}}{6}$ into $\frac{\partial f}{\partial t}=0$ we get the equation

$$\frac{1}{6}\left(t+\sqrt{t}\right)\left(1+\frac{1}{2\sqrt{t}}\right)-6t=0$$

that is

$$t+\frac{3}{2}\sqrt{t}+\frac{1}{2}=36t$$

or

$$70\left(\sqrt{t}\right)^2-3\sqrt{t}-1=0$$

which has the solution

$$\sqrt{t} = \frac{3 \pm \sqrt{9+4\cdot 70}}{2\cdot 70} = \frac{3\pm\sqrt{289}}{140}$$
$$= \frac{20}{140} = \frac{1}{7}$$

since we must have $\sqrt{t} > 0$. At this value we have $q = \frac{8}{49\cdot 6} = \frac{4}{3\cdot 49}$ and $f(q,t) = \frac{7}{3\cdot 49^2} = \frac{1}{3\cdot 343} > 0$, so this is indeed the maximum.

3. Constrained optimization

(8) (MATH 105 final, 2017) Use the method of Lagrange Multipliers to find the maximum value of the utility function $U = f(x,y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x,y) = 50x + 100y - 500,000 = 0$, where $x \geq 0$ and $y \geq 0$.

Solution: If $x = 0$ or $y = 0$ we have $f(x,y) = 0$ while if $x, y > 0$ we have $f(x,y) > 0$ so the maximum must be in the interior of the domain (and occur at a critical point). By the method of Lagrange Multipliers the maximum occurs at a point x, y where

$$\begin{cases} 4x^{-3/4}y^{3/4} = 50\lambda \\ 12x^{1/4}y^{-1/4} = 100\lambda \\ 50x + 100y - 500,000 = 0\,. \end{cases}$$

If $\lambda = 0$ then either $x = 0$ or $y = 0$ by the first two equations, which isn't the case, so $\lambda \neq 0$ and we can divide the second equation by the first. We get:

$$3\frac{x}{y} = 2\,,$$

that is $3x = 2y$. Writing the equation of the constraint as $x + 2y = 10,000$ we see that we must have $4x = 10,000$ so $x = 2,500$ and $y = \frac{3x}{2} = 3,750$. Since this is the only solution it must be the maximum, and the value is

$$f(2500, 3750) = 16\cdot\left(\frac{10^4}{4}\right)^{1/4}\left(3\frac{10^4}{8}\right)^{3/4}$$
$$= 2^4\frac{10}{\sqrt{2}}\cdot 10^3\cdot 3^{3/4}\cdot 2^{-9/4} = 10^4\cdot 2^{\frac{5}{4}}\cdot 3^{3/4}$$
$$= 20,000\times 2^{1/4}3^{3/4}\,.$$

(9) Labour-Leisure model: a person can choose to spend L hours a day not working ("leisure"), working $24 - L$ hours with way w. Suppose their fixed income is V dollars per day. Their consumption of goods is them $C = w(24 - L) + V$, equivalenly $C + wL = 24w + V$ (here C, L are variables while w, V are constants). If their utility function is $U = U(C, L)$ find a system of equations for maximum utility.

Solution: We need to maximize $U(C,L)$ subject to the *budget constraint* $C + wL = 24w + V$, so we get the system

$$\begin{cases} \frac{\partial U}{\partial C} & = \lambda \\ \frac{\partial U}{\partial L} & = \lambda w \\ C + wL & = 24w + V\,. \end{cases}$$

4. Combination problems

(10) Find the maximum and minimum values of $f(x,y) = -x^2 + 8y$ in the disc $R = \left\{x^2 + y^2 \leq 25\right\}$.

Solution: $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = 8$, so f has no critical points in the interior of the disc (or anywhere, for that matter), and the minimum and maximum must occur on the boundary, where

$x^2+y^2=25$ or equivalently $G(x,y)=x^2+y^2-25=0$. By the method of Lagrange multipliers the extrema on the boundary occur where

$$\begin{cases} -2x &= \lambda(2x) \\ 8 &= \lambda(2y) \\ x^2+y^2 &= 25\,. \end{cases}$$

We can rewrite the first equation $2x(\lambda+1)=0$ so either $x=0$ or $\lambda=-1$. In the first case we have $y=\pm 5$ and the function values are $f(0,5)=40$, $f(0,-5)=-40$. In the second case the second equation reads $8=-2y$ so $y=-4$ and $x=\pm\sqrt{25-16}=\pm 3$. At these points we have

$$\begin{aligned} f(\pm 3,-4) &= -9+8\cdot(-4) \\ &= -41\,. \end{aligned}$$

The maximum of f is therefolre 40, attained at $(0,5)$, and the minimum is -41, attained at the two points $(\pm 3,-4)$.

(11) (MATH 105 final, 2015) Find the maximum and minimum values of $f(x,y)=(x-1)^2+(y+1)^2$ in the disc $R=\left\{x^2+y^2\le 4\right\}$.

Solution: We have $\frac{\partial f}{\partial x}=2(x-1)$ and $\frac{\partial f}{\partial y}=2(y+1)$ so the only critical point is $(1,-1)$ where $f(1,-1)=0$. Since $f(x,y)\ge 0$ for all x,y this must be the global minimum. The maximum must therefore occur on the boundary where $x^2+y^2=4$ so let $G(x,y)=x^2+y^2-4$. By the method of Lagrange multipliers the maximum on the boundary will occur at a point where

$$\begin{cases} 2(x-1) &= \lambda(2x) \\ 2(y+1) &= \lambda(2y) \\ x^2+y^2 &= 4 \end{cases}.$$

Since $x=0$ does not solve the first equation we can divide by $2x$ to get $1-\frac{1}{x}=\lambda$ and similarly from the second equation we get $1+\frac{1}{y}=\lambda$. Subtracting these two equations gives $\frac{1}{x}+\frac{1}{y}=0$ so $y=-x$. Plugging this into the constraint we get $2x^2=4$ so $x=\pm\sqrt{2}$ and there are two *constrained critical points* on the boundary, $\left(\sqrt{2},-\sqrt{2}\right)$ and $\left(-\sqrt{2},\sqrt{2}\right)$. We have

$$\begin{aligned} f\left(\sqrt{2},-\sqrt{2}\right) &= \left(\sqrt{2}-1\right)^2+\left(-\sqrt{2}+1\right)^2 = 2\left(\sqrt{2}-1\right)^2 = 2\left(3-2\sqrt{2}\right) \\ f\left(-\sqrt{2},\sqrt{2}\right) &= \left(-\sqrt{2}-1\right)^2+\left(\sqrt{2}+1\right)^2 = 2\left(\sqrt{2}+1\right)^2 = 2\left(3+2\sqrt{2}\right) \end{aligned}$$

so clearly $\left(\sqrt{2},-\sqrt{2}\right)$ is the minimum of f *on the boundary* and $f\left(-\sqrt{2},\sqrt{2}\right)$ is the maximum of f *on the boundary*, which (as we saw before) was the maximum on the domain. In summary: the minimum value of f on R is 0, attained at $(1,-1)$, and the maximum value is $6+4\sqrt{2}$, attained at $\left(-\sqrt{2},\sqrt{2}\right)$.