Lior Silberman's Math 412: Problem Set 4 (due 10/2/2023)

Practice

M1. Let U,V be vector spaces and let $U_1 \subset U$, $V_1 \subset V$ be subspaces.
(a) "Naturally" embed $U_1 \otimes V_1$ in $U \otimes V$.
(b) Is $(U \otimes V)/(U_1 \otimes V_1)$ isomorphic to $(U/U_1) \otimes (V/V_1)$?

M2. Let $(\cdot,\cdot)$ be a non-degenerate bilinear form on a finite-dimensional vector space U, defined by the isomorphism $g\colon U \to U'$ via $(\underline{u},\underline{v}) \stackrel{\text{def}}{=} (g\underline{u})(\underline{v})$.
(a) For $T \in \text{End}(U)$ define $T^\dagger = g^{-1}T'g$ where T' is the dual map. Show that $T^\dagger \in \text{End}(U)$ satisfies $(\underline{u},T\underline{v}) = \left(T^\dagger\underline{u},\underline{v}\right)$ for all $\underline{u},\underline{v} \in V$.
(b) Show that $(TS)^\dagger = S^\dagger T^\dagger$.
(c) Show that the matrix of $T^\dagger$ wrt an $(\cdot,\cdot)$-orthonormal basis is the transpose of the matrix of T in that basis.

The dual map

1. Let U,V,W be vector spaces, and let $T \in \text{Hom}(U,V)$, and let $S \in \text{Hom}(V,W)$.
(a) (The abstract meaning of transpose) Suppose U,V be finite-dimensional with bases $\{\underline{u}_j\}_{j=1}^m \subset U$, $\{\underline{v}_i\}_{i=1}^n \subset V$, and let $A \in M_{n,m}(F)$ be the matrix of T in those bases. Show that the matrix of the dual map $T' \in \text{Hom}(V',U')$ with respect to the dual bases $\left\{\underline{u}'_j\right\}_{j=1}^m \subset U'$, $\{\underline{v}'_i\}_{i=1}^n \subset V'$ is the transpose ${}^{\text{t}}A$.
(b) Show that $(ST)' = T'S'$. It follows that ${}^{\text{t}}(AB) = {}^{\text{t}}B\,{}^{\text{t}}A$.

Bilinear forms

In problems 2,3 we assume 2 is invertible in F and fix F-vector spaces V,W.

2. (Alternating pairings and symplectic forms) Let V,W be vector spaces, and let $[\cdot,\cdot] : V \times V \to W$ be a bilinear map.
(a) Show that $(\forall \underline{u},\underline{v} \in V : [\underline{u},\underline{v}] = -[\underline{v},\underline{u}]) \leftrightarrow (\forall \underline{u} \in V : [\underline{u},\underline{u}] = 0)$ (Hint: consider $\underline{u}+\underline{v}$).
DEF A form satisfying either property is *alternating*. We now suppose $[\cdot,\cdot]$ is alternating.
PRAC Show that the *radical* $R = \{\underline{u} \in V \mid \forall \underline{v} \in V : [\underline{u},\underline{v}] = 0\}$ of the form is a subspace.
(b) The form $[\cdot,\cdot]$ is called *non-degenerate* if its radical is $\{\underline{0}\}$. Show that setting $[\underline{u}+R,\underline{v}+R] \stackrel{\text{def}}{=} [\underline{u},\underline{v}]$ defines a non-degenerate alternating bilinear map $(V/R) \times (V/R) \to W$.
RMK Note that you need to justify each claim, starting with "defines".

3. (Darboux's Theorem) Suppose now that V is finite-dimensional, and that $[\cdot,\cdot] : V \times V \to F$ is a non-degenerate alternating form.
DEF The *orthogonal complement* of a subspace $U \subset V$ is a set $U^\perp = \{\underline{v} \in V \mid \forall \underline{u} \in U : [\underline{u},\underline{v}] = 0\}$.
PRAC Show that $U^\perp$ is a subspace of V.
(a) Show that the restriction of $[\cdot,\cdot]$ to U is non-degenerate iff $U \cap U^\perp = \{\underline{0}\}$.
(*b) Suppose that the conditions of (a) hold. Show that $V = U \oplus U^\perp$, and that the restriction of $[\cdot,\cdot]$ to $U^\perp$ is non-degenerate.

(c) Let $\underline{u} \in V$ be non-zero. Show that there is $\underline{u}' \in V$ such that $[\underline{u}, \underline{u}'] \neq 0$. Find a basis $\{\underline{u}_1, \underline{v}_1\}$to $U = \mathrm{Span}\{\underline{u}, \underline{u}'\}$ in which the matrix of $[\cdot,\cdot]$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(d) Show that $\dim_F V = 2n$ for some n, and that V has a basis $\{\underline{u}_i, \underline{v}_i\}_{i=1}^n$ in which the matrix of $[\cdot,\cdot]$ is block-diagonal, with each 2×2 block of the form from (d).

RECAP Only even-dimensional spaces have non-degenerate alternating forms, and up to choice of basis, there is only one such form.

Tensor products

4. For finite-dimensional U, V construct a natural isomorphism $\mathrm{End}(U \otimes V) \to \mathrm{Hom}(U, U \otimes \mathrm{End}(V))$.

   SUPP Generalize this to a natural isomorphism $\mathrm{Hom}(U \otimes V_1, U \otimes V_2) \to \mathrm{Hom}(U, U \otimes \mathrm{Hom}(V_1, V_2))$.

5. Let U, W be vector spaces with U finite-dimensional, and let $A \in \mathrm{Hom}(U, U \otimes W)$. Given a basis $\{\underline{u}_j\}_{j=1}^{\dim U}$ of U let $\underline{w}_{ij} \in W$ be defined by $A\underline{u}_j = \sum_{i=1}^{\dim U} \underline{u}_i \otimes \underline{w}_{ij}$ and define $\mathrm{Tr}_U A = \sum_{i=1}^{\dim U} \underline{w}_{ii} \in W$. Show that this definition is independent of the choice of basis.

6. (Partial traces) Let U, V be real vector spaces equipped with non-degenerate inner products.
   (a) Show that $\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{U\otimes V} \stackrel{\mathrm{def}}{=} \langle u_1, u_2 \rangle_U \langle v_1, v_2 \rangle_V$ induces an inner product on $U \otimes V$.
   (b) Let $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$. Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ ($A^\dagger, B^\dagger$ are defined in M2).
   (c) Let $P \in \mathrm{End}(U \otimes V)$, interpreted as an element of $\mathrm{Hom}(U, U \otimes \mathrm{End}(V))$ as in 4. Show that $(\mathrm{Tr}_U P)^\dagger = \mathrm{Tr}_U (P^\dagger)$.
   (*d) [Thanks to J. Karczmarek] Let $\underline{w} \in U \otimes V$ be non-zero, and let $P_{\underline{w}} \in \mathrm{End}(U \otimes V)$ be the orthogonal projection on $\underline{w}$. It follows from (c) that $\mathrm{Tr}_U P_{\underline{w}} \in \mathrm{End}(V)$ and $\mathrm{Tr}_V P_{\underline{w}} \in \mathrm{End}(U)$ are both selfadjoint. Show that their non-zero eigenvalues are the same.

Supplementary problems

A. (Extension of scalars) Let $F \subset K$ be fields. Let V be an F-vectorspace.

(a) Considering K as an F-vectorspace (see PS1), we have the tensor product $V_K = K \otimes_F V$ (the subscript means "tensor product as F-vectorspaces"). For each $\alpha \in K$ set $\alpha(x \otimes \underline{v}) \stackrel{\text{def}}{=} (\alpha x) \otimes \underline{v}$. Show that this extends to an F-linear map $K \otimes_F V \to K \otimes_F V$ giving V_K the structure of a K-vector space. This construction is called "extension of scalars"

(b) Let $\{\underline{v}_i\}_{i \in I} \subset V$ be a set of vectors. Show that it is linearly independent (resp. spanning) iff $\{1_K \otimes \underline{v}_i\}_{i \in I} \subset V_K$ is linearly independent (resp. spanning). Conclude that $\dim_K (K \otimes_F V) = \dim_F V$.

(c) Let $V_N = \mathrm{Span}_{\mathbb{R}} \left(\{1\} \cup \{\cos(nx), \sin(nx)\}_{n=1}^{N} \right)$. Then $\frac{d}{dx} \colon V_N \to V_N$ is not diagonable. Find a different basis for $\mathbb{C} \otimes_{\mathbb{R}} V_N$ in which $\frac{d}{dx}$ is diagonal. Note that the elements of your basis are not "pure tensors", that is not of the form $af(x)$ where $a \in \mathbb{C}$and $f = \cos(nx)$ or $f = \sin(nx)$.

B. DEF: An *F-algebra* is a triple $(A, 1_A, \times)$ such that A is an F-vector space, $(A, 0_A, 1_A +, \times)$ is a ring, and (compatibility of structures) for any $a \in F$ and $x, y \in A$ we have $a \cdot (x \times y) = (a \cdot x) \times y = x \times (a \cdot y)$. Because of the compatibility from now on we won't distinguish the multiplication in A and scalar multiplication by elements of F.

(a) Verify that $\mathbb{C}$ is an $\mathbb{R}$-algebra, and that $M_n(F)$ is an F-algebra for all F.

(b) More generally, verify that if R is a ring, and $F \subset R$ is a subfield then R has the structure of an F-algebra. Similarly, that $\mathrm{End}_F(V)$ is an F-algebra for any vector space V.

(c) Let A, B be F-algebras. Give $A \otimes_F B$ the structure of an F-algebra.

(d) Show that the map $F \to A$ given by $a \mapsto a \cdot 1_A$ gives an embedding of F-algebars $F \hookrightarrow A$.

(e) (Extension of scalars for algebras) Let K be an extension of F. Give $K \otimes_F A$ the structure of a K-algebra.

(f) Show that for V finite-dimensional, $K \otimes_F \mathrm{End}_F(V) \simeq \mathrm{End}_K(K \otimes_F V)$.

C. The *center* $Z(A)$ of a ring is the set of elements that commute with the whole ring.

(a) Show that the center of an F-algebra is an F-subspace, containing the subspace $F \cdot 1_A$.

(b) Show that the image of $Z(A) \otimes Z(B)$ in $A \otimes B$ is exactly the center of that algebra.