Lior Silberman's Math 412: Problem Set 6 (due 3/3/2023)

1. Let $U \in M_n(F)$ be *strictly upper-triangular*, that is upper triangular with zeroes along the diagonal. Show that $U^n = 0$ and construct such U with $U^{n-1} \neq 0$.

2. Let V be a finite-dimensional vector space, $T \in \mathrm{End}(V)$.

   (*a) Show that the following statements are equivalent:

   (1) $\forall \underline{v} \in V : \exists k \geq 0 : T^k \underline{v} = \underline{0}$;   (2) $\exists k \geq 0 : \forall \underline{v} \in V : T^k \underline{v} = \underline{0}$.

   DEF A linear map satisfying (2) is called *nilpotent*. Example: see problem 1.

   SUPP For any infinite-dimensional V find an example of $T \in \mathrm{End}(V)$ satisfying (1) but not (2). Such maps are called *locally nilpotent*.

   (b) Find nilpotent $A, B \in M_2(F)$ such that $A + B$ isn't nilpotent.

   (c) Suppose that $A, B \in \mathrm{End}(V)$ are nilpotent and that A, B commute. Show that $A + B$ is nilpotent.