Lior Silberman's Math 535, Problem Set 1b: Analysis

Haar measure

Let X be a locally compact topological space. Write $C(X)$ for the space of continuous real-valued functions on X, and for $f \in C(X)$ write $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. It is well-known that the subspace $C_b(X) = \{f \in C(X) \mid \|f\|_\infty < \infty\}$ is complete in the supremum norm and that it contains the subspace $C_c(X)$ of compactly supported functions.

DEFINITION. A Radon *measure* on X is a linear functional $\mu\colon C_c(X) \to \mathbb{C}$ such that $\mu(f) \geq 0$ if $f \geq 0$ (that is, if $f(x) \in \mathbb{R}_{\geq 0}$ for each x). If μ is a Radon measure and $f \in C_c(X)$ we often write $\int f\,d\mu$ instead of $\mu(f)$.

1. (Preliminaries)
   (a) Show that the closure of $C_c(X)$ in $C_b(x)$ is the space $C_0(X)$ of functions vanishing at infinity (continuous functions f such that for all $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ if $x \notin K$.
   (b) Let $X' \subset X$ and let μ be a Radon measure on X. Show that $\mu \restriction_{C_c(X')}$ is a Radon measure on X'.
   (c) In particular, suppose Y is compact. Show that a Radon measure on Y is a bounded linear functional on $C(Y) = C_b(Y) = C_c(Y)$.
   (d) Conclude that a Radon measure is continuous for the *direct limit topology* on $C_c(X)$.

2. Let G be a locally compact topological group.
   (a) Let $f, f' \in C_c(G)$ be non-negative, and let $U \subset G$ be open. Set

   $$(f : U) = \inf\left\{\sum_{i=1}^{n} \alpha_i \mid \alpha_i \geq 0, f \leq \sum_{i=1}^{n} \alpha_i \cdot 1_{g_i U}\right\}.$$

   Show that $0 \leq (f : U) < \infty$. Assuming $f' \neq 0$ show that $(f : U) \leq (f' : U)(f : f')$ for an appropriately defined $(f : f')$ which is independent of U.
   (b) Let $\mathcal{N}$ be the set of open neighbourhoods of the identity in G; for $U \in \mathcal{N}$ set $F_U = \{V \in \mathcal{N} \mid V \subset U\}$. Show that $\mathcal{F} = \{S \subset \mathcal{N} \mid \exists U : S \supset F_U\}$ is a filter on $\mathcal{N}$ (that is, if $S_1, S_2 \in \mathcal{F}$ and $T \subset \mathcal{N}$ then $S_1 \cap S_2, S_1 \cup T \in \mathcal{F}$). Show that for any $V \in \mathcal{N}$ there is $S \in \mathcal{F}$ with $V \notin S$ ("$\mathcal{F}$ is not contained in any principal filter"). Let $\omega \subset \mathcal{N}$ be a maximal filter containing $\mathcal{F}$.
   (c) Fix $f_0 \in C_c(G)$ which is non-negative and non-zero. Show that $\mu(f) \stackrel{\text{def}}{=} \lim_{U \to \omega} \frac{(f : U)}{(f_0 : U)}$ extends to a G-invariant Radon measure on G. Such μ is called a (left) *Haar measure* on G.
   (d) Show that $\mu(f) > 0$ for all non-negative non-zero $f \in C_c(G)$.
   (e) Suppose G is non-compact. Show that μ is an *infinite measure*: that $\mu\colon C_c(X) \to \mathbb{C}$ is unbounded with respect to the supremum norm.

3. (Uniqueness of Haar measure) Let μ_1, μ_2 be a left Haar measure on G. Fix a compact neighbourhood K of the identity in G.
   (a) Given $f \in C_c(G)$ show that f is *uniformly continuous:* for any $\varepsilon > 0$ there is an open subset $U \subset K$ such that for all $x \in G$, $u \in U$ we have $|f(xu) - f(x)| < \varepsilon$.
   (b) Let $\chi \in C_c(U)$ be positive such that $\mu(\chi) = 1$ and let $(f \star \chi)(x) = \int_G f(xu)\chi(u)\,\mathrm{d}\mu_1(u)$. Show that $\|f \star \chi - f\|_\infty \leq \varepsilon$ and hence
$$\left| \int \mathrm{d}\mu_2(x) \int \mathrm{d}\mu_1(u) f(xu)\chi(u) - \int \mathrm{d}\mu_2(x) f(x) \right| \leq \varepsilon \mu_2(K)\,.$$
   (c) Changing variables on the LHS show that
$$\left| \int \mathrm{d}\mu_2(x) f(x) - E \int \mathrm{d}\mu_1(x) f(x) \right| \leq \varepsilon \mu_2(K)$$
   with $E = \int \chi(x^{-1}) d\mu_2(x) > 0$. Conclude that μ_1, μ_2 are proportional.

4 Fix a left Haar measure μ on G.
   (a) For $f \in C_c(G)$ and $g \in G$ let $(R_g f)(x) = f(xg)$ be the left regular representation. Show that $\mu_g(f) \stackrel{\text{def}}{=} \mu(R_g f)$ is also a left Haar measure on G. It follows that there is $\delta_G(g) \in \mathbb{R}^\times_{>0}$ such that $\mu_g(f) = \delta_G(g^{-1})\mu(f)$ for all f.
   RMK The g^{-1} is so that $\mu(Ag) = \delta_G(g)\mu(A)$ for every left Haar measure μ, measurable $A \subset G$ and $g \in G$.
   (b) Show that $\delta_G \colon G \to \mathbb{R}^\times_{>0}$ is a continuous group homomorphism and is independent of the choice of Haar measure.
   DEF The map $\delta_G \colon G \to \mathbb{R}^\times_{>0}$ is called the *modular character* of G. The group G is called *unimodular* if δ_G is the trivial character (identically 1).
   (c) Show that $\mu\left(f(x^{-1})\delta(x)\right)$ is a right Haar measure on G. Conclude that G is unimodular iff every left Haar measure is a right Haar measure.
   (d) Suppose G is compact. Show that $\mathrm{Hom}_{\mathrm{cts}}\left(G, \mathbb{R}^\times_{>0}\right) = \{1\}$ and conclude that G is unimodular.
   (e) Show that every abelian group and every discrete group is unimodular.

5. (Example of Haar measure) Let $\mathrm{GL}_n(\mathbb{R}) = \{g \in M_n(\mathbb{R}) \mid \det g \neq 0\}$. Let μ be the measure on $\mathrm{GL}_n(\mathbb{R})$ with density $\frac{1}{|\det(g)|^n}$ wrt Lebesgue measure – in other words:
$$\int f(g)\,\mathrm{d}\mu(g) = \iint f\left((g_{ij})_{i,j=1}^n\right) \frac{1}{|\det(g)|^n} dg_{11} \cdots dg_{nn}\,.$$
   Show that μ is a left- and right-invariant Haar measure.

The inverse and implicit function theorems

6. Let $U \subset \mathbb{R}^{n+m}$ be open, and let $f \in C^1(U;\mathbb{R}^m)$. Write $df(x,y) = (K,J)$ where $K(x,y) \colon \mathbb{R}^n \to \mathbb{R}^m$ and $J(x,y) \colon \mathbb{R}^m \to \mathbb{R}^m$ and suppose that at some $(x_0,y_0) \in U$ we have that $J(x_0,y_0)$ is invertible.
   (a) Show that there exists neighbourhoods V of x_0 and W of y_0 such that $V \times W \subset U$ and such that for all $x \in V$ there there is a unique $y \in W$ such that $(x,y) \in U$ and $f(x,y) = f(x_0,y_0)$. *Hint*: show that if x is close enough to x_0 then the approximate Newton iteration
   $$H_x(y) = y - (J(x_0,y_0))^{-1} \cdot (f(x,y) - f(x_0,y_0))$$
   is a well-defined contraction on some neighbourhood of y_0, hence converges to a unique fixed point.
   DEF Let $g(x)$ is the unique fixed point of part (a). We say $g \colon V \to W$ is the function *defined implicitely* by $f(x,y) = f(x_0,y_0)$. In particular we have $f(x, g(x)) = f(x_0,y_0)$.
   (b) Show that $g \in C^1(V)$, and that if V is small enough we have $dg = -(J(x,g(x)))^{-1} K(x,g(x))$
   (c) State the *inverse function theorem*, which the special case $n = 0$ of the result of (a),(b).