Math 535, Lecture 13, 27/2/2023

Last time: G ctd cpt Lie gp, $T \subset G$ max'l torus
$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{t} = \mathrm{Lie}\, T$. $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$

Let T act on $\mathfrak{g}_{\mathbb{C}}$ via adjoint rep'n. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad \text{where:}$$

(1) $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$, (2) if $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{g}_\alpha$ then

$e(z) = e^{2\pi i z}$

$$\mathrm{Ad}(\exp(H)) \cdot X_\alpha = e(\alpha(H)) \cdot X_\alpha \quad (*)$$

$\alpha \in \Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subseteq \mathfrak{t}^*$ where $\Lambda = \mathrm{Ker}(\exp_T)$

complex conjugation in $(*)$ gives: $\overline{X_\alpha} \in \mathfrak{g}_{-\alpha}$
(Cor: $-\Phi = \Phi$)

Lemma: $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta \Rightarrow [X_\alpha, X_\beta] \in \mathfrak{g}_{\alpha+\beta}$

Thm: $\mathrm{rk}\, G \overset{\text{def}}{=} \dim_{\mathbb{R}} \mathfrak{t} = 1 \Rightarrow G \simeq SO(3)$ or $SU(2)$
($\mathfrak{g} \simeq \mathfrak{su}(2)$, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}_2 \mathbb{C}$)

Pf: Given any $\beta \in \Phi$ let $X_\beta \in \mathfrak{g}_\beta$, know $\bar{X}_\beta \in \mathfrak{g}_{-\beta}$ ($\bar{X}_{-\beta}$)
set $H_\beta = [X_\beta, X_{-\beta}] \neq 0$ since max'l torus is Id.

$$\bar{H}_\beta = [X_{-\beta}, X_\beta] = -[X_\beta, X_{-\beta}] = -H_\beta$$

so $H_\beta \in i t_{\mathbb{C}}$, $iH_\beta \in t_{\mathbb{R}} = t$.

Fix $H \in t$ non-zero. Every real root α is determined by $\alpha(H)$, order the roots by the values $\alpha(H)$. Let β be the smallest positive root. Choose X_β, $X_{-\beta}$ as above

Set $V = \mathbb{C} X_{-\beta} \oplus t_{\mathbb{C}} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \subset \mathfrak{g}_{\mathbb{C}}$

Observe: if $\gamma \in \Phi^+$ $(\gamma > 0)$, $X \in V$, $X_\gamma \in \mathfrak{g}_\gamma$ then $[X_\gamma, X] \in V$:

if $X \in \mathfrak{g}_\alpha$ then $[X_\gamma, X_\alpha] \in \mathfrak{g}_{\alpha+\gamma} \subset V$ since $\alpha+\gamma > \alpha > 0$

If $X \in t_{\mathbb{C}}$, $[X_\gamma, X] \in \mathfrak{g}_\gamma \subset V$ (since $\gamma > 0$)

$[X_\gamma, X_{-\beta}] \in \mathfrak{g}_{\alpha-\beta}$, $\alpha - \beta \geq 0$ since β smallest positive root.

<u>Particular case:</u> V is $\mathrm{ad}(X_\beta)$-stable.

<u>Also:</u> V is $\mathrm{ad}(X_{-\beta})$-inv't.

<u>Pf:</u> $\mathrm{ad}(X_{-\beta})\cdot X_{-\beta} = 0$

$\mathrm{ad}(X_{-\beta})\cdot t_{\mathbb{C}} \subseteq \mathbb{C}\cdot X_{-\beta}$

$\mathrm{ad}(X_{-\beta})\cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha-\beta} \subset V$ ($\uparrow$ $\alpha-\beta \geq 0$)

$\mathrm{ad}_X \cdot Y = [X, Y]$

Consider $\mathrm{ad}(H_\beta)$ where $H_\beta = [X_\beta, X_{-\beta}] \in t_{\mathbb{C}}$.

<u>Saw:</u> ad is a Lie alg. rep'n so $\mathrm{ad}(H_\beta) = [\mathrm{ad}X_\beta, \mathrm{ad}X_{-\beta}]$

so V is H_β-stable and $\mathrm{Tr}_{\mathbb{C}}(H_\beta|V) = 0$

On the other hand by the def'n of V,

$$\mathrm{Tr}(H_\beta|V) = -2\pi i\rho(H_\beta) + 0 + \sum_{\alpha>0} \dim_{\mathbb{C}} \mathfrak{g}_\alpha \cdot 2\pi i\alpha(H_\beta)$$

divide by 2π, get:

$$(\dim_{\mathbb{C}} \mathfrak{g}_\beta - 1)\,\beta(iH_\beta) + \sum_{\alpha > \beta} (\dim_{\mathbb{C}} \mathfrak{g}_\alpha)\cdot \alpha(iH_\beta) = 0$$

Now $iH_\beta \in \mathfrak{t}_{\mathbb{R}}$ is non-zero so $\beta(iH_\beta)$, $\alpha(iH_\beta)$ all are non-zero & have same sign (either iH_β is a positive multiple of H used to define order, or a negative multiple).

$\Rightarrow$ all terms are zero, i.e. no roots $\alpha > \beta$
$\dim \mathfrak{g}_\beta = 1$

$\Rightarrow \dim \mathfrak{g}_\beta = \mathfrak{g}_{-\beta} = 1$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{-\beta} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_\beta$

has $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = 3$, $\Rightarrow \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} \leq 3$, so $\mathfrak{g} = \mathfrak{su}(2)$ and $G \cong SU(2)$ or $SO(3)$. ∎

The algebraic Weyl group

Back to general (ctd) G. Take $\alpha \in \Phi$, let

$$U_\alpha = \ker(\alpha) \subset \mathfrak{t}_{\mathbb{R}}.$$

$$G_\alpha = Z_G(U_\alpha)$$

Recall the exponential root $\chi_\alpha(\exp H) = e(\alpha(H))$

See $H \in \mathfrak{u}_\alpha \Rightarrow \exp H \in \ker \chi_\alpha$, so $\mathfrak{u}_\alpha = \text{Lie} \ker(\chi_\alpha)$

So $\exp(\mathfrak{u}_\alpha)$ is a closed subtorus of T of codimension 1.

Example: $G = SU(2)$ $T = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$

$$\mathfrak{t} = \mathbb{R} \cdot \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad \Lambda = 2\pi\mathbb{Z} \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} = \begin{pmatrix} 0 & e^{2i\theta} \\ & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \right] = 2i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 2\pi i & \\ & -2\pi i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \right] = 4\pi i$$

$$\alpha \begin{pmatrix} 2\pi i & \\ & -2\pi i \end{pmatrix} = 2.$$

$$\chi_\alpha \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = e^{2i\theta} \qquad \ker(\chi_\alpha) = \{\pm I\}.$$

Back to general case. $\mathfrak{u}_\alpha = \ker(\alpha)$

$G_\alpha = Z_G(\mathfrak{u}_\alpha)$

T commutative $\Rightarrow T \leq Z_G(\mathfrak{u}_\alpha)$

so T max'l torus of G_α. But $Z_\alpha = \exp(\mathfrak{u}_\alpha)$ is central in G_α, so G_α / Z_α has rk 1

Prop: G_α is a ctd subgp of semisimple rank 1.

Also (1) $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$

(2) $W(G_\alpha : T) \cong C_2$

(3) Let $s_\alpha \in W(G_\alpha ; T) \subset W(G : T)$

be the non-trivial element. Then $Ad(s_\alpha)|_{\mathfrak{t}} \in GL(\mathfrak{t})$ is a reflection in the hyperplane $\mathfrak{u}_\alpha$

Pf: Saw: Z_G(Lie alg. of torus) is connected, rk $G_\alpha / Z_\alpha = 1$

G_α non-commutative: $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ commute with $\mathfrak{u}_\alpha$:

if $H \in \mathfrak{u}_\alpha$, $X_\alpha \in \mathfrak{g}_\alpha$, $[H, X_\alpha] = \alpha(H) X_\alpha = 0$

so $\mathfrak{u}_\alpha$ commute with $Re(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$

which has dim ≥ 2, does not commute with $\mathfrak{t}$.

Set $\bar{G}_\alpha = G_\alpha / Z_\alpha$, $\bar{T}_\alpha = T / Z_\alpha$ set: $\bar{G}_\alpha$ of rk 1, max'l torus $\overline{T}_\alpha$. By thm $\overline{G}_\alpha \cong SU(2)$ or $SO(3)$

(i) Let β be a root proportional to α.
Then $\pm\beta(H) = 0$ if $H \in \mathfrak{u}_\alpha$ so

$$Re(\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}) \in \text{lie } G_\alpha$$

the sum of all these (over $\beta \in \mathbb{R}\alpha$) is direct, disjoint from $\mathfrak{u}_\alpha$. ⇒ it injects into lie $G_\alpha/\mathfrak{u}_\alpha$

complexify: $\mathfrak{g}_\beta$ inject into $(\text{lie } G_\alpha)_\mathbb{C}/(\mathfrak{u}_\alpha)_\mathbb{C}$.
But $\text{lie } G_\alpha/\mathfrak{u}_\alpha = \text{lie } \overline{G}_\alpha$ so $\dim_\mathbb{R} \text{lie } \overline{G}_\alpha = 3$

so $(\text{lie } G_\alpha)_\mathbb{C}/(\mathfrak{u}_\alpha)_\mathbb{C} = (\text{lie } \overline{G}_\alpha)_\mathbb{C} \cong \overline{\mathfrak{t}}_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

& only roots prop to α are $\pm\alpha$, $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ 1d

<u>aside</u>: why is $\overline{\mathfrak{t}}_\alpha = \mathfrak{t}/\mathfrak{u}_\alpha$ a max'l torus of lie $\overline{G}_\alpha$?

<u>Pf</u>: if $\overline{S} \subset \text{lie } \overline{G}_\alpha$ is a max'l torus $\supseteq \overline{\mathfrak{t}}_\alpha$
its inverse image in $\mathfrak{g}$ is a subalgebra S
st $[S,S] \subseteq \mathfrak{u}_\alpha \subseteq Z(\text{lie } G_\alpha)$

$\Rightarrow$ if $X \in \mathfrak{s}$ $ad_X|_{\mathfrak{s}}$ is nilpotent,

$ad_X(\mathfrak{s}) \subseteq \mathfrak{u}_\alpha$ so $(ad_X)^2(\mathfrak{s}) = 0$

But in a cpt gp $X \in$ torus $\Rightarrow$ ad_X is diagonable so its eigenvalues are 0, so $ad_X|_{\mathfrak{s}} = 0$ so $\mathfrak{s}$ is commutative, so S is a torus of G containing t, so $S = t$.