Math 535, lecture 30 , 24/3/2023

Last time: $W^\lambda = \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{t}_\mathbb{C} \oplus \mathfrak{a})} \mathbb{C}^\lambda$

$\lambda \in \mathfrak{t}_\mathbb{C}^*$ $\qquad L^\lambda = W^\lambda / \text{sum of proper submodules.}$

Thm: (1) L^λ unique irred module with highest weight λ,
(2) If λ is dominant and algebraically integral then $\dim_\mathbb{C} L^\lambda < \infty$.

State Thm: Suppose further that λ is integral. Then L^λ integrates to a representation of G.

Today†: Character χ_λ of L^λ.

If π is a f.d. rep'n of G, the character of π is
$$\chi_\pi(g) = \mathrm{tr}(\pi(g) | V_\pi)$$

Facts (Peter-Weyl): (1) χ_π is a class function:

$$\chi_\pi(gxg^{-1}) = \chi_\pi(x)$$

(2) If σ, π irred, non-isomorphic, $\chi_\sigma \perp \chi_\pi$ in $L^2(G)$

(χ_π is a sum of matrix coefficients of π)

(3) $\|\chi_\pi\|_{L^2(G)} = 1$ if π is irred

(4) $\{\chi_\sigma\}_{\sigma \in \hat{G}} \subset L^2(G)$ is an o.n.b of the subspace of class functions

<u>Goal</u>: Formula for χ_λ in terms of λ.
In particular for $\chi_\lambda(e) = \dim_{\mathbb{C}} L^\lambda$.

<u>Today</u>: Preliminaries.
<u>Observation</u>: By conjugacy of max'l tori, we saw that

$$G/\mathrm{Ad}(G) \simeq T/W$$

$\Rightarrow \chi_\lambda$ is a W-inv't function on T, determined by those values.

also $\chi_\lambda(\exp H) = \sum_{\mu} \dim V_\mu \cdot e(\mu(H))$
$\mathfrak{t}^{\mathfrak{a}}$

and $\dim V_{w\mu} = \dim V_\mu$

$\Rightarrow \quad \chi_\lambda \in \mathbb{Z}[\{e\circ\mu\}_{\mu \in t^*}]$

<u>Preliminary 1:</u> Weyl Integration formula
(relate integration in G and T)

<u>Preliminary 2:</u> The ring of characters.

<u>The Weyl integration formula</u>

The map

$$q : G/T \times T \to G$$
$$(g, t) \mapsto g^{-1} t g$$

is surjective, G-equivariant, $\#W$-to-1 restricted to $G/T \times T_{reg}$

$T_{reg} = T \setminus \text{walls}$

Sinular set has codim 1 in T & in G, has measure 0, so regular set is open, dense, of full measure.

<u>Lemma:</u> The Jacobian determinant of q at (e, t) is $J(t) = \det(\mathrm{Id} - \mathrm{Ad}_{t^{-1}} | \mathfrak{g}/\mathrm{Lie}\,T)$

Pf: Compute $dq(e,t)$.

Let $X \in \mathfrak{g}/\mathrm{Lie}\,T$, $H \in \mathrm{Lie}\,T$.

to 1st order

$$q(e+X, t(1+H)) \approx (Id - X)\, t\, (Id + H)(Id + X)$$

$$\approx t - Xt + tX + tH$$

so

$$q(e+X, t(1+H)) - t = t(X + H - t^{-1}Xt)$$

$$\Rightarrow dq_{(e,t)}(X+H) = (X+H) - Ad_{t^{-1}} \cdot X$$

$$= Id_{T_{(e,t)}(G/T \times T)} - Ad_{t^{-1}} \cdot X$$

$\Rightarrow J(t) = \det(\text{derivative}) =$

$$= \det\left((Id_{\mathrm{Lie}G/\mathrm{Lie}T} - Ad_{t^{-1}} \mid \mathrm{Lie}G/\mathrm{Lie}T) \oplus Id_{\mathrm{Lie}T}\right)$$

$$= \det(I - Ad_{t^{-1}} \mid \mathrm{Lie}\,G/\mathrm{Lie}\,T).$$

For $t \in T_{reg}$ this is to formula:

$$J(\exp H) = \prod_{\alpha \in \Phi} (1 - e(\alpha(H))).$$

Cor: (Weyl integration formula)

$$\boxed{\#W \int_G f(g)\,dg = \int_T dt\, J(t) \int_G dg\, f(g^{-1}tg)}$$

Cor: Let $f \in C(G)$ be a class function. Then

$$\int f(g)\,dg = \frac{1}{\#W} \cdot \int_T f(t) J(t)\, dt.$$

Example: orthogonality of characters takes form

$$\int_T \overline{\chi_\lambda(t)}\, \chi_{\lambda'}(t)\, J(t)\, dt = \delta_{\lambda,\lambda'}.$$

Algebra: The ring $\mathbb{Z}[\{e^{\mu}\}_{\mu \in t^*}]$

Let $I \subset t^*$ be an additive subgp, set

$$R_I = \mathbb{Z}[\{e^{\mu}\}_{\mu \in I}]$$

(as a ring of functions on $t = \mathrm{Lie}\, T$)

<u>Observation:</u> For any <u>finite</u> $A \subset I$, the subgp $\langle A \rangle \subset I$ is torsion-free hence free & finite rank, so of the form $\langle B \rangle$ where $B \subset \mathfrak{t}^*$ is linearly indep over $\mathbb{Q}$ $\Rightarrow$

$$R_{\langle A \rangle} = R_{\langle B \rangle} \cong \mathbb{Z}[\{x_i^{\pm}\}_{i=1}^{\#B}]$$

where $x_i \leftrightarrow e^{\mu_i}$, $\{\mu_i\}_{i=1}^{\#B} = B$

So $R_{\langle B \rangle}$ is a localization of $\mathbb{Z}[\{x_i\}_{i=1}^{\#B}]$

$\mathbb{Z}$ UFD $\Rightarrow$ $\mathbb{Z}[\{x_i\}]$ UFD $\Rightarrow$ $\mathbb{Z}[\{x_i^{\pm}\}]$ UFD

$\Rightarrow$ direct limit $R_I = \bigcup_A R_{\langle A \rangle}$ is a UFD

(we will take $I = \{\mu \in \mathfrak{t}^* \mid \mu(\Gamma) \subset \mathbb{Z}\}$, the integral weights)