Math 535, Lecture 32 , 29/3/2023

Real semisimple Groups

Groups like $SL_n(\mathbb{R})$, $PGL_n(\mathbb{R}) = GL_n(\mathbb{R})/Z$.

$$O(n,1) - \mathrm{Isom}(\mathbb{H}^{(n)})$$

Start with semisimple Lie algebras

$\mathfrak{g}$ = Lie algebra / k field, $\mathrm{char}(k)=0$.

Def: The Killing form is the bilinear form on $\mathfrak{g}$

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}X \cdot \mathrm{ad}Y \,|\, \mathfrak{g})$$

(symmetric since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$)

Lemma: This form is $\mathfrak{g}$-invariant:

$\mathrm{ad}_X W = [X, W]$

$$B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$$

(if $\mathfrak{g} = \mathrm{Lie}\, G$ this is the infinitesimal version of G-invariance: $B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X, Y)$)

<u>Lemma:</u> V f.d. k-vsp, $T \in \mathrm{End}_k(V)$, $T(V) \subset U \subset V$ U subspace Then $T|_U : U \to U$ makes sense and

$$\mathrm{Tr}_k(T|U) = \mathrm{Tr}_k(T|V).$$

<u>Pf:</u> Extend basis of U to V.

Cor: Let $\mathfrak{a} \subset \mathfrak{g}$ be a Lie ideal. Then if $X, Y \in \mathfrak{a}$ we have

$$B_{\mathfrak{a}}(X, Y) = B_{\mathfrak{g}}(X, Y)$$

(image of $\mathrm{ad}_X \mathrm{ad}_Y$ lies in $\mathfrak{a}$)

<u>Cor:</u> Let $\mathfrak{a} \subset \mathfrak{g}$ be an abelian ideal. Then $\mathfrak{a} \subset \mathrm{rad}\, B$.

<u>Pf:</u> Let $X \in \mathfrak{g}$, $Y \in \mathfrak{a}$. The image of ad_Y is contained in $\mathfrak{a}$, which is ad_X-inv't.

So $\mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y | \mathfrak{g}) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y | \mathfrak{a}) = 0$ since $\mathrm{ad}_Y|_{\mathfrak{a}} = 0$. So $B(X, Y) = 0$ if $Y \in \mathfrak{a}$

Def: Call $\mathfrak{g}$ semisimple if $B_{\mathfrak{g}}$ is non-degenerate ($\mathrm{rad}(B)=0$).

Fix a s.s. Lie alg. $\mathfrak{g}$, for a subset $\mathfrak{a}\subset\mathfrak{g}$ write

$$\mathfrak{a}^{\perp}=\{X\in\mathfrak{g}\mid \forall Y\in\mathfrak{a}: B(X,Y)=0\}.$$

Prop: Let $\mathfrak{a}\subset\mathfrak{g}$ be a Lie ideal. Then $\mathfrak{a}^{\perp}$ is an ideal and $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{a}^{\perp}$.

Pf: ① Let $X\in\mathfrak{a}^{\perp}$, $Z\in\mathfrak{g}$ want $[X,Z]\in\mathfrak{a}^{\perp}$. For any $Y\in\mathfrak{a}$, $[Z,Y]\in\mathfrak{a}$ so $[Z,Y]\perp X$.

$$\Rightarrow B([Z,X],Y)=B(\mathrm{ad}_Z X,Y)=-B(X,\mathrm{ad}_Z Y)=0$$

so $[Z,X]\in\mathfrak{a}^{\perp}$.

② Next $[\mathfrak{a},\mathfrak{a}^{\perp}]=\{0\}$: let $X\in\mathfrak{a}$, $Y\in\mathfrak{a}^{\perp}$. For all $Z\in\mathfrak{g}$,

$$B(Z,[X,Y])=-B(\underset{\substack{\uparrow\\ \mathfrak{a}}}{\mathrm{ad}_X Z},Y)=0$$

Since B is non-degenerate, $[X,Y]=0$

③ $\mathfrak{a} \cap \mathfrak{a}^\perp$ is a commutative ideal. Since $\mathrm{rad}(\mathfrak{g}) = 0$
$\Rightarrow \mathfrak{a} \cap \mathfrak{a}^\perp = \{0\}$

④ $\mathrm{rad}\, B_{\mathfrak{a}} = \mathfrak{a} \cap \mathfrak{a}^\perp = \{0\}$, so $B_{\mathfrak{a}}$ is non-degen.

Let $X \in \mathfrak{g}$. Have $Y \in \mathfrak{a}$ s.t. $\forall Z \in \mathfrak{a}$,

$$B_{\mathfrak{g}}(X, Z) = B_{\mathfrak{a}}(Y, Z) = B_{\mathfrak{g}}(Y, Z)$$

(the linear functional $\{Z \mapsto B_{\mathfrak{g}}(X,Z)\} \in \mathfrak{a}^*$ must be reperentable using $B_{\mathfrak{a}}$)

$\Rightarrow B_{\mathfrak{g}}(X-Y, Z) = 0$ for all $Z \in \mathfrak{a}$.

$\Rightarrow X = Y + (X-Y) \in \mathfrak{a} \oplus \mathfrak{a}^\perp$.

<u>Cor</u>: Every s.s. Lie alg. is the direct sum of simple ideals.

<u>Real semisimple groups</u>

Let $\mathfrak{g}$ be a s.s. Lie alg. Consider $\mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$

closed in analytic topology, also Zariski-closed.

In particular $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) \subset \mathrm{Lie}(GL(\mathfrak{g})) = \mathrm{End}_{\mathbb{R}\text{-}v.sp}(\mathfrak{g})$

Every ad_X exponentiates to an automorphism so $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$, $\mathrm{ad}_X \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$

__Prop:__ $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$ is the image of $\mathfrak{g}$ by adjoint representation

(Cor: $\mathrm{Out}(\mathfrak{g}) = \pi^0(\mathrm{Aut}(\mathfrak{g}))$)

__Remark:__ $\mathrm{Aut}(\mathbb{R}^n) = GL_n(\mathbb{R})$, so really need $\mathfrak{g}$ s.s.

Pf:

Let $\mathfrak{h} = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$. Since $Z_{\mathfrak{g}} = \{0\}$, the image in $\mathfrak{h}$ of ad is isomorphic to $\mathfrak{g}$

For $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$, $\mathrm{ad}_{[Y,X]} = [Y, \mathrm{ad}\, X]$

commutator in $\mathfrak{gl}(\mathfrak{g})$

$\Rightarrow \mathrm{ad}\,\mathfrak{g} \subset \mathfrak{h}$ is a Lie ideal.

$\Rightarrow (\mathrm{ad}\,\mathfrak{g})^{\perp} \subset \mathfrak{h}$ is also an ideal (wrt $B_{\mathfrak{h}}$)

$\Rightarrow (\mathrm{ad}\,\mathfrak{g}) \cap (\mathrm{ad}\,\mathfrak{g})^{\perp} \subset \mathrm{rad}\, B_H \subset \mathrm{rad}\, B_{\mathrm{ad}\mathfrak{g}} = \{0\}$

$\uparrow$ $\mathrm{ad}\,\mathfrak{g} \subset H$ is an ideal

$\Rightarrow$ (same proof) $[\mathrm{ad}\,\mathfrak{g}, (\mathrm{ad}\,\mathfrak{g})^{\perp}] = 0$

so sum

$$\mathrm{ad}\,\mathfrak{g} \oplus (\mathrm{ad}\,\mathfrak{g})^{\perp} \text{ is direct.}$$

$\Rightarrow$ any $D \in (\mathrm{ad}\,\mathfrak{g})^{\perp}$ acts trivially on $\mathfrak{g}$.

But D is an aut of $\mathfrak{g}$ so $D = 0$, i.e. $(\mathrm{ad}\,\mathfrak{g})^{\perp} = 0$

(The Lie subgp of $\mathrm{Aut}(\mathfrak{g})$ corresponding to $(\mathrm{ad}\,\mathfrak{g})^{\perp}$ acts trivially on $\mathfrak{g}$) $\blacksquare$

Def: Call a Lie group G semisimple if $\mathfrak{g} = \mathrm{Lie}\, G$ is semisimple.

Cor: Let G be a ctd s.s. Lie gp. Then
$$\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})^{\circ}.$$

$\Rightarrow \mathrm{Ad}(G)$ closed in $\mathrm{Aut}(\mathfrak{g})$, hence in $GL(\mathfrak{g})$.

HW: If G is cpt, ctd, finite centre then

$B_{\mathfrak{g}}$ is negative definite.

Thm: Let G be ctd, s.s., finite centre, suppose $B_{\mathfrak{g}}$ is negative definite. Then G is cpt.

Pf: $G/Z(G) = Ad(G)$ is a closed subgp of the orthogonal group $O(B_{\mathfrak{g}}) \subset GL(\mathfrak{g})$ which is cpt if $B_{\mathfrak{g}}$ is definite.

Thm: Let $X \in \mathfrak{g}$. Then there exist unique commuting $X_s, X_n \in \mathfrak{g}$ s.t. ad_{X_s} is s.s. (= diagonable / $\mathbb{C}$)
ad_{X_n} is nilpotent
and $X = X_s + X_n$.

Pf: $Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ is Zariski-closed,
$Lie(Aut(\mathfrak{g})) = ad\, \mathfrak{g} = \mathfrak{g}$,
claim follows from Jordan decomp in alg. gp.