Math 535, Lecture 38, 1/5/2024

Recently: Iwasawa decomposition

Today: Infinite-dimensional representations

Summary of structure theory:

G s.s. Lie gp, Cartan involution $\Theta \in \mathrm{Aut}(G)$:
$\theta = d\Theta \in \mathrm{Aut}(\mathfrak{g})$ s.t. $B_\theta(X,Y) = B(X, \theta Y)$ is neg. def.

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ $\quad \mathfrak{k} = \{X \mid \theta X = X\}$, $\mathfrak{p} = \{X \mid \theta X = -X\}$

with:

$K = G^\Theta = \{g \in G \mid \Theta(g) = g\}$. Then $\mathrm{Lie}\, K = \mathfrak{k}$,
contains $Z = Z(G)$, K/Z cpt, max'l cpt in G/Z.
and

$G \cong K \times \exp \mathfrak{p}$. (polar decomp)

$\mathfrak{a}$ = max'l abelian subalg in $\mathfrak{p}$. (ad-diagonalizable)

restricted roots $\Sigma = \left\{ \begin{array}{l} \text{nonzero chars} \\ \text{of } \mathfrak{a} \text{ on } \mathfrak{g} \end{array} \right\}$

Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, $\quad \mathfrak{g}_0 = \mathfrak{a} \oplus \underbrace{Z_k(\mathfrak{a})}_{\mathfrak{m}}$

As before divide Σ into pos/neg roots set $\mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \Theta \mathfrak{n} = \sum_{\alpha < 0} \mathfrak{g}_\alpha$ then

$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$, $G \cong N \times A \times K$ where N, A are subgps with lie alg $\mathfrak{n}, \mathfrak{a}$

A normalizes N.

Set $M = Z_K(A)$, Lie $M = \mathfrak{m}$

Set $P_0 = NAM$ "minimal parabolic subgp".

<u>Example:</u> $G = SL_n(\mathbb{R})$, $\Theta(g) = {}^t g^{-1}$, $K = O(n)$

A = diagonal, pos entries = exp (diag.)
N = upper-triangular unipotent.

$M = \mathrm{diag}(\pm 1, \dots, \pm 1)$

<u>Example:</u> $G = SL_2(\mathbb{C})$, $\Theta(g) = \overline{{}^t g^{-1}}$, $K = SU(2)$
$A = \{\mathrm{diag}(e^{t/2}, e^{-t/2})\}$, $N = \{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{C}\}$, $M = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\}$

Have symmetric space $S = G/K \cong NA$ ← "Iwasawa co-ords on S".

Representations of G

We saw: a cts rep'n of G is a pair (π, V) where V is a TVS, $\pi : G \times V \to V$ is a cts action by linear maps.

Assume V locally convex, quasi-complete, so that if X cpt, $f : X \to V$ cts, μ on X Radon measure then

$$\int_X f(x)\, d\mu(x) \quad \text{exists}$$

(Sufficient to assume V is a Banach space or Frechet)

Truth: rep'n theory of G actually algebraic, choice V doesn't matter.

Usually G acts on X, so on functions on X, usually doesn't quite matter if we take $L^2(X)$, $C_c^\infty(X)$, $L^p(X)$, ...

<u>Def</u>: Call $v \in V$ <u>smooth</u> if $g \mapsto \pi(g)v$ is smooth as a function $G \to V$. V^∞: smooth vectors

($f : X \to V$ is diff if $f(x+h) = f(x) + L \cdot h + O(|h|^2)$ with $L : T_x X \to V$ cts).

<u>Recall</u>: for $f \in C_c(G)$, define $\pi(f)v = \int_G f(g)\, \pi(g) v \, dg$ (integral wrt Haar measure).

<u>Thm</u>: (Gårding) Let $f \in C_c^\infty(G)$. Then $\pi(f)v \in V^\infty$.

<u>Cor</u>: $V^\infty \subset V$ is dense.

<u>Pf</u>: Observe $\pi(h) \cdot \pi(f) v = \int_G f(g)\, \pi(hg) v \, dg$

$$= \int_G f(h^{-1}g)\, \pi(g) \, dg = \pi(L_h f) \cdot v$$

diff under integral sign gives for $X \in \mathfrak{g}$

$X \cdot (h \mapsto \pi(h)\pi(f)v)\big|_{h=1} = \pi(L_X f) \cdot v$

(ie $\{\pi(f)\underline{v} \mid \underline{v}\in V, f\in C_c^\infty\}$ is closed under diff)

so contained in V^∞. Also dense:

if $W\subset V$ convex nbd of $\underline{v}\in V$, $\mathcal{U}\subset G$ nbd of 1 st if $g\in\mathcal{U}$, $\pi(g)\underline{v}\in W$, then if $f\in C_c^\infty(\mathcal{U})$, $f\geq 0$, $\int f=1$, then by convexity $\pi(f)\underline{v}\in W$,

so $\left\{\pi(f)\underline{v} \;\middle|\; \begin{matrix} f\in C_c^\infty(G) \\ \underline{v}\in V\end{matrix}\right\}$ is dense

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<u>Examples</u> $\mathbb{R}$ act on $L^2(\mathbb{R})$ by translation

$\varphi\in L^2(\mathbb{R})$ <u>smooth</u>: function $t\mapsto (x\mapsto \varphi(t+x))$ is smooth $\mathbb{R}\to L^2(\mathbb{R})$

diff $\Leftrightarrow$ $\varphi'\in L^2$, twice diff $\Leftrightarrow$ $\varphi''\in L^2$, ..

Sobolev inequalities $\Rightarrow$ φ smooth ($\in C^\infty(\mathbb{R})$)

(Here $C_c^\infty(\mathbb{R})\subset L^2(\mathbb{R})$, is $\mathbb{R}$-inv't, dense).

Assume now G s.s.

<u>Recall</u>: $\underline{v} \in V$ is K-<u>finite</u> if $\underline{v}$ is contained in a f.d. K-inv't subspace of V.

<u>Know</u>: K-finite vectors are dense in any cts rep'n of K (Peter-Weyl)

In particular, $(C_c^\infty(G))_K$ is dense in $C_c^\infty(G)$

If $f \in C_c^\infty(G)$ is K-finite, so is $\pi(f)\underline{v}$ since

$$\text{span}\{\pi(k)\pi(f)\underline{v}\}_{k\in K} = \{\pi(\tilde{f})\underline{v} \mid \tilde{f} \in \text{span}\{L_k f\}_{k\in K}\}$$

<u>Conclusion</u>: $V_K^\infty = \{\underline{v} \in V \mid \begin{matrix} v \text{ is smooth} \\ K\text{-finite} \end{matrix}\} \subset V$ is dense

<u>Observe</u>: K acts on V_K^∞ by $\pi|_{V_K^\infty}$.

But G doesn't: if $\underline{v} \in V_K$, $\pi(g)\underline{v}$ is gKg^{-1}-finite

However $\mathfrak{g}$, $\mathcal{U}(\mathfrak{g})$ do act, since $\mathfrak{g}$ is a f.d. K-rep'n

so if $\underline{v}$ is K-finite, smooth, $\pi(X)\underline{v}$ is also K-finite, smooth:

$\pi(k)\,\pi(X)\,\pi(k^{-1}) = \pi(\mathrm{Ad}_k \cdot X)$ on V^∞.

if $W \subset V$ f.d., K-invt

so map $\mathfrak{g} \times W \to V$: $(X, \underline{w}) \to \pi(X)\underline{w}$

is bilinear, extends to K-hom $\mathfrak{g} \otimes W \to V$

so image is f.d., contains $\pi(X)\,\underline{w}$.

Def: (Harish-Chandra) A $(\mathfrak{g}, K)$-module is a vsp W which is simultaneously a $\mathcal{U}(\mathfrak{g})$-module and a rep'n of K where $W_K = W$, compatible in that

1. $k \cdot X \cdot k^{-1} \cdot \underline{w} = (\mathrm{Ad}_k X) \cdot \underline{w}$ for all $\underline{w} \in W$, $k \in K$, $X \in \mathfrak{g}$.

2. Write action of K via π, then if $X \in \mathrm{Lie}(K)$ then $\pi(X)\,\underline{w} = d\pi(X)\,\underline{w}$

Def: Call $(\pi, V), (\pi', V') \in \mathrm{Rep}_{cts}(G)$ infinitesimally equivalent if $V^\infty_K \cong V'^\infty_K$ as $(\mathfrak{g}, K)$-modules

Idea: Classify irred $(\mathfrak{g}, K)$-modules then ask which integrate to rep'ns of type ...