Math 535, Lecture 39, 5/5/2023

Last time: ① G Lie group, (π, V) rep'n on "reasonable" TVS, then
$$V^\infty = \{ v \in V \mid (g \mapsto \pi(g)v) \in C^\infty(G;V) \}$$
is dense

(Pf: if $f \in C_c^\infty(G)$, $v \in V$, then $\pi(f)v \in V^\infty$)

$\mathfrak{g}$, $U(\mathfrak{g})$ act on V^∞ by $\pi(X)v = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tX))v$.

compatibly with G-action

② G s.s., $K < G$ max'l cpt subgp then $(C_c^\infty(G))_K \subset C_c(G)$ is dense (Peter-Weyl) $\Rightarrow$
$$V_K^\infty = \{ \pi(f)v \mid {v \in V \atop f \in C_c^\infty(G)_K} \} \subset V$$
is dense

(can also use $V_K^\infty = \{ \pi(h)v \mid {v \in V^\infty \atop h \in C(K)_K} \}$)

V_K^∞ is a Harish-Chandra module / $\mathfrak{g}$-K module: carries compatible actions of K, $\mathfrak{g}$.

<u>Today</u>: basics of $\mathfrak{g}$;K modules, $SL_2(\mathbb{R})$

<u>Def</u>: Call (π, V), (σ, W) infinitesimally equivalent if V_K^∞, W_K^∞ are isomorphic as $\mathfrak{g}$-K modules

<u>Observe</u>: If $W \subset V$ is closed, G-inv't, then $W_K^\infty \subsetneq V_K^\infty$ (have distinct closures)

$\Rightarrow$ If V_K^∞ is irred (has no proper submodules $\neq 0$)

then V is irred (has no G-inv't closed subspaces)

<u>Fact</u>: converse also true: if V irred, so is V_K^∞.

<u>Prop</u>: (Schur's lemma) let V be an irred $\mathfrak{g}$-K module. Then $\mathrm{End}_{\mathfrak{g},K}(V) \cong \mathbb{C}$.

<u>Pf</u>: Step 1: $\dim_{\mathbb{C}} V$ is countable

observe $\mathfrak{g}$ is a rep'n of K

$\Rightarrow T(\mathfrak{g}) = T(\mathfrak{g})_K \Rightarrow \mathcal{U}(\mathfrak{g})$ consists of K-finite vectors

Now let $W \subset V$ be a f.d. K-inv't subspace

Endow $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} W$ with $\mathfrak{g}$-K module structure by letting $\mathfrak{g}$ act on $\mathcal{U}(\mathfrak{g})$, K act by

$$k \cdot (X \otimes w) = \mathrm{Ad}_k \cdot X \otimes \pi(k) \underline{w}$$

action of K on $\mathcal{U}(\mathfrak{g})$

informally $k \cdot X \cdot \underline{w} = k X k^{-1} \cdot k \underline{w}$

π gives hom $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} W \to V$

$$X \otimes w \mapsto \pi(X) \underline{w}$$

Image contains W, so by irred is V.

$$\dim_{\mathbb{C}} (\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} W) = \underset{\aleph_0}{\underbrace{\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{g})}} \cdot \underset{\text{finite}}{\underbrace{\dim_{\mathbb{C}} W}} < \aleph_0 .$$

<u>Step 2:</u> $\mathrm{End}_{\mathfrak{g},K} V$ is a division algebra.

<u>Pf:</u> Let $T: V \to V$ be a hom of $(\mathfrak{g}, K)$-modules. Then $\ker T$, $\mathrm{Im}\, T$ are submodules so if $T \neq 0$, $\ker T = \{0\}$, $\mathrm{Im}(T) = V$, so T invertible

(purely algebraic theory: no worry whether τ^{-1} is bounded)

Step 3: let $\tau \in \mathrm{End}_{q,k}(V)$. Either $\tau = \lambda \cdot \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$, or $\tau - \lambda$ are all invertible.

consider $\{(\tau-\lambda)^{-1}\}_{\lambda \in \mathbb{C}}$.

Fact: $\{\frac{1}{z-\lambda}\}_{\lambda \in \mathbb{C}} \subset \mathbb{C}(z)$ are linearly indep $/\mathbb{C}$.

(if λ_i distinct $a_i \neq 0$ $\sum_i \frac{a_i}{z-\lambda_i}$ has poles at λ_i) is nonzero

$\Rightarrow$ $\underline{v} \in V$ nonzero $\{(\tau-\lambda)^{-1}\underline{v}\}_{\lambda \in \mathbb{C}} \subset V$ indep:

If $\sum_i a_i \cdot (\tau - \lambda_i)^{-1}\underline{v} = 0$ then $\sum_i a_i (\tau - \lambda_i)^{-1}$ is not invertible, so is the zero endomorph.

Contradiction: $\dim V \leq \aleph_0$, can't contain a continuum of indep vectors

<u>Cor:</u> An irred $(\mathfrak{g}, K)$-module supports at most one inv't inner pdt.

(inv't: K acts unitarily, $\mathfrak{g}$ acts anti-hermitian

$$\langle \pi(k) v, \pi(k) v \rangle = \langle v, k \rangle$$

$$\langle \pi(X) u, v \rangle + \langle u, \pi(X) v \rangle = 0.$$
)

<u>Pf:</u> inv't inner pdt $\Leftrightarrow$ hom $V \to V_K'$ unique up to scaling by same argument.

<u>Cor:</u> Classifying unitary dual $\hat{G}$ (isom classes of irred unitary rep'ns on Hilbert space)

$\Leftrightarrow$ ① classify irred $(\mathfrak{g}, K)$-modules
("admissible dual")

② determine which are unitarizable

__Examples__ $SL_2(\mathbb{R})$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathfrak{g} = \operatorname{span}\{H, X_+, X_-\} \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$W = X_+ - X_-$$

$$\mathfrak{a} = \mathbb{R}\cdot H, \quad \mathfrak{n} = \mathbb{R}\cdot X_+, \quad \mathfrak{k} = \mathbb{R}\cdot W$$

$$\exp(tW) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\mathfrak{g} = \mathfrak{sl}_2\mathbb{R} = \mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$$

Let K act on $\mathfrak{g}$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\cdot\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\cdot\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2 t - \sin^2 t & -2\cos t\sin t \\ -2\cos t \sin t & -(\cos^2 t - \sin^2 t) \end{pmatrix} = \cos(2t)\cdot H - \sin(2t)(X_+ + X_-)$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & -s \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} cs & c^2 \\ -s^2 & -sc \end{pmatrix} = cs\cdot H + c^2 X_+ - s^2 X_-$$

In $\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$ co-ords:

$$k(\theta) = \exp(W\cdot\theta)$$

$\mathrm{Ad}(k(\theta))\cdot H =$

$$\frac{1}{2}(e^{2i\theta}+e^{-2i\theta})\cdot H - \frac{i}{2}(e^{2i\theta}-e^{-2i\theta})W + i(e^{2i\theta}-e^{-2i\theta})X_+$$

$\mathrm{Ad}(k(\theta))X_+ =$

$$\frac{-i}{4}(e^{2i\theta}-e^{-2i\theta})\cdot H - \frac{1}{4}(e^{2i\theta}-2+e^{-2i\theta})W + \frac{1}{2}(e^{2i\theta}+e^{-2i\theta})X_+$$

$\mathrm{Ad}(k(\theta))\cdot W = W$ (k commutative)

Set $J_+ = H + iX_+ + iX_- = H + 2iX_+ - iW$.

$$\mathrm{Ad}(k(\theta))\cdot J_+ = e^{2i\theta}H + 2i e^{2i\theta}X_+ - ie^{2i\theta}W$$
$$= e^{2i\theta}\cdot J_+$$

Set $J_- = H - iX_+ - iX_-$ then

$$\mathrm{Ad}(k(\theta))\cdot J_- = e^{-2i\theta}J_-.$$

<u>summary:</u> $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$, in $\mathfrak{g}_{\mathbb{C}}$ found J_+, J_-.

Let V be an irred $(\mathfrak{g}, K)$-module for $SL_2(\mathbb{R})$

Let $v_n \in V$ be a vector s.t. $k(\theta) \cdot v = e^{in\theta} v$

then $k(\theta) \cdot (J_+ v_n) = e^{2i\theta} \cdot J_+ \cdot k(\theta) \cdot v_n$

$$= e^{(2+n)i\theta} J_+ v_n$$

so $J_+ v_n$ has weight $n+2$

similarly $J_- v_n$ " " $n-2$

Finally, $J_- J_+ = (H - iX_+ - iX_-)(H + iX_+ + iX_-)$

$$= H^2 + X_+^2 + X_-^2 + i[H, X_+] + i[H, X_-]$$

$$+ X_+ X_- + X_- X_+$$

$$= \underbrace{H^2 + X_+^2 + X_-^2 + X_+ X_- + X_- X_+}_{\Omega} + 2iW$$

$\Omega \in Z(\mathcal{U}(sl_2 \mathbb{R}))$ check : $[\Omega, H] = [\Omega, X_+]$
$= [\Omega, X_-] = 0$

so in irred $(\mathfrak{g}, K)$-mod. Ω = scalar

$\Rightarrow$ If Ω has ev. λ in V, then

$$J_- \cdot J_+ v_n = \lambda \cdot v_n - 2n\, v_n = (\lambda - 2n) v_n$$

$W \cdot V_n = in$

(similar if $J_+ J_-$)

<u>Cor</u>: The module V is exactly

$$\operatorname{span}\{J_+^m v_n\}_{m \geq 0} \cup \{v_n\} \cup \{J_-^m v_n\}_{m \geq 0}$$

(excluding zeroes)

<u>Conclusion</u>: in an irred rep'n of $SL_2(\mathbb{C})$, the weights are an interval

$$n_{min},\ n_{min}+2,\ n_{min}+4,\ \dots,\ n_{max}$$

where all have same parity, maybe $n_{min} = -\infty$, $n_{max} = \infty$

each weights occurs with mult. 1

<u>conversely</u>: use these to define $\mathfrak{g}, K$ module.

When is $J_+ v_n = 0$? then $J_- J_+ v_n = \underline{0}$

$v_n \neq 0$

so $(\lambda - 2n) v_n = \underline{0}$

so if λ is an even integer, get $v_{\lambda/2}$ is highest-weight.

$$J_- J_+ = \Omega + 2iW$$

so if λ is an even integer, $\underline{v}_{-\lambda/2}$ is lowest-weight.

<u>Conclusion</u>: classification of irred $(\mathfrak{g}, K)$-modules for $SL_2(\mathbb{R})$:

① $\lambda \notin 2\mathbb{Z}$ Then get two such modules:

span $\{v_n\}_{n \text{ even}}$, span $\{v_n\}_{n \text{ odd}}$.

either case, $\begin{cases} J_+ v_n = v_{n+2} \\ J_- v_n = J_- J_+ (v_{n-2}) = (\lambda - 2(n-2)) v_{n-2} \\ W v_n = in\, v_n \end{cases}$

② $\lambda = 2m$, $m \in \mathbb{Z}$

get several: (1) $\mathrm{Span}\{v_n\}_{n \not\equiv m \,(2)}$
same action as above

(2) $m \geq 0$ $\mathrm{Span}\{v_n\}^{-m \leq n \leq m}_{n \equiv m\,(2)}$

(exactly f.d. rep'n from earlier)
& dim $m+1$

② $m \geq 0$ $\mathrm{Span}\{v_{m+2}, v_{m+4}, v_{m+6}, \dots\}$

rules as above (except $J_- v_{m+2} = 0$)

③ $m \geq 0$, $\mathrm{Span}\{v_{-m-2}, v_{-m-4}, \dots\}$

except $J_+(v_{-m-2}) = 0$)

④ $m < 0$

Can even determine unitarity: v_n always ⊥
want J_3 anti-Hermitian want
$(J_\pm)^\dagger = (H + iX_+ + iX_-)^\dagger = -H + iX_+ + iX_- = -J_-$

so $J_- J_+ = - J_+^\dagger J_+$ need to have ≤ 0 ev:

need $\lambda - 2n \leq 0$ for all n in rep'n space

need λ real, ~

Can calc inner pdt since

$$\langle v_{n+2}, v_{n+2} \rangle = \langle J_+ v_n, J_+ v_n \rangle$$

$$= \langle v_n, J_+^\dagger J_+ v_n \rangle = -\langle v_n, J_- J_+ v_n \rangle$$

$$= -(\lambda - 2n) \langle v_n, v_n \rangle$$

concrete "by hand" point of view:
generically

$$V = \operatorname{span}\{v_n\}_{\substack{n \equiv 0 \\ \text{or} \\ n \equiv 1}} \quad (2)$$

$\mathfrak{g}$ acts by J_+, J_-, W

Next time: induced rep'ns

__Cor:__ Every irrep of $SL_2(\mathbb{R})$ is admissible in that every K-type occurs finitely many times (0 or 1 times)