1A)

3. The Derivative (20/9/2023)

Goals.

(1) The derivative at a point
(2) Tangent lines & linear approximations
(3) The derivative as a function

Last Time.

Limits: $\lim_{x \to a} f(x) = L$ means "as x approaches a, the values of f get closer to L".

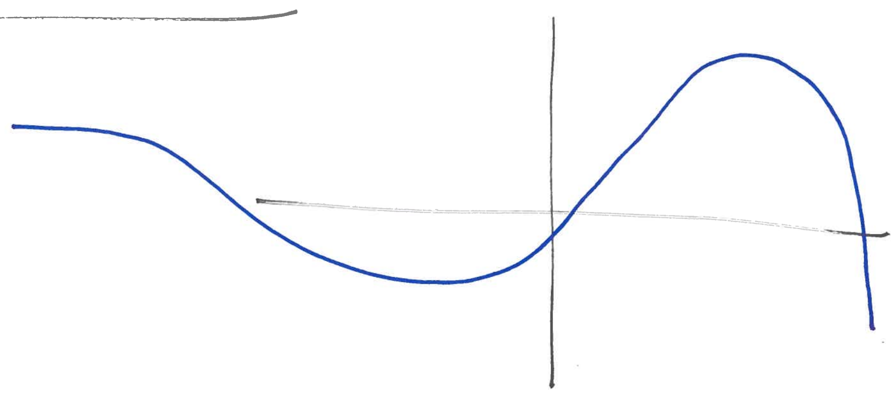
$f(a)$ **immaterial** – only look at x close to a, $x \neq a$.

Extended sense: $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) ← **blowup vertical**

Asymptotics: $\lim_{x \to \infty} f(x) = L$ (or $x \to -\infty$) **asymptote**

horizontal asymptote

Derivatives



Key fact: For many functions, if you "zoom in" towards a point on graph, the graph increasingly looks like a straight line

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WS1 (a), (b)

Saw: ① $\lim_{x \to 2} \frac{\Delta y}{\Delta x} = 4$ (geometry)

② $f(2+h) - f(2) \sim 4h$ (asymptotics)

$\Leftrightarrow f(2+h) \approx f(2) + 4h = 4 + 4h$

want slope here, can compute $\frac{f(2+h) - f(2)}{h} \sim \frac{4h}{h} = 4$

Definition: Let f be defined at & near $x = a$. The **derivative** of f at a is the limit

$$\frac{df}{dx}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(say f is **differentiable** at a if limit exists)

Math 100A – WORKSHEET 3
THE DERIVATIVE

1. THREE VIEWS OF THE DERIVATIVE

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2, 4)$ is a point on the graph of $y = f(x)$.

(a) Let (x, x^2) be another point on the graph, close to $(2, 4)$. What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?

slope is $\frac{\Delta y}{\Delta x} = \frac{x^2-4}{x-2} = x+2 \xrightarrow[x\to 2]{} 4$

change of variable → $x=2+h$, $h=x-2$

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2+h)$ as $h \to 0$? What about $f(2+h) - f(2)$?

2^2

$f(2+h) = (2+h)^2 = 4 + 4h + h^2 \sim 4 = 2^2 = f(2)$

"how does $f(2+h)$ approach $f(2)$?" = "how does $f(2+h) - f(2)$ approach 0"

$f(2+h) - f(2) = 4h + h^2 \sim 4h$

(c) What is $\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$?

Also 4.

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2, 4)$?

Date: 20/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

One more p.o.v.:

Saw: $f(2+h) - f(2) \sim 4h$

If we "wiggle" x-coord by h, the y-coord "wiggles" by $4h$

("velocity" point of view)

Ws 1(d):

if we zoom into $f(x) = x^2$ at $x = 2$, see a line of slope 4 (in the limit)

Also, line passes through $(2,4)$

$\Rightarrow$ the line is

$$y = 4x - 4$$

$$y - 4 = 4(x-2)$$

also "linear approx" to $f(x)=x^2$ at $x=2$ $\rightarrow$ $y = 4 + 4(x-2)$

$f(2)$ $f'(2)$ point 2

(2) $\star\star$ An enzymatic reaction occurs at rate $k(T) = T(40 - T) + 10T$ where T is the temperature in degrees celsius. The current temperature of the solution is 20°C. Should we increase or decrease the temperature to increase the reaction rate?

2. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a+h) \approx f(a) + f'(a)h$

(3) Find $f'(a)$ if

(a) ⋆ $f(x) = x^2$, $a = 3$.

$$f(3+h) = (3+h)^2 = 9 + 6h + h^2 \approx 9 + 6h$$

so $f'(3) = 6$

$$\lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} (6+h) = 6$$

(b) ⋆⋆ $f(x) = \frac{1}{x}$, any a.

$$f(a+h) = \frac{1}{a+h} = \frac{1}{a} + \left(\frac{1}{a+h} - \frac{1}{a}\right) = \frac{1}{a} + \frac{a-(a+h)}{a(a+h)}$$

$$= \frac{1}{a} - \frac{1}{a(a+h)} h \approx \frac{1}{a} - \frac{1}{a^2} h$$

as $h \to 0$ $\frac{h}{a(a+h)} \sim \frac{h}{a^2}$

so $f'(a) = -\frac{1}{a^2}$

Takeaway: usually f' defined for most of domain of f, can think of f' as function

(c) ⋆⋆ $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

(4) ⋆⋆ Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h)-\cos 5}{h}$, $\lim_{x\to 0} \frac{\sin x}{x}$

derivative of $\cos\theta$ at $\theta = 5$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x-0} = (\sin\theta)'\Big|_{\theta=0}$$

(5) ⋆⋆⋆ (Final, 2015, variant – gluing derivatives) Is the function

$$f(x) = \begin{cases} x^2 & x \le 0 \\ x^2 \cos \frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?

check: $\frac{f(x)}{x} \underset{x \to 0}{\to} 0$ so $f'(0) = 0$, $f'(0)$ exists

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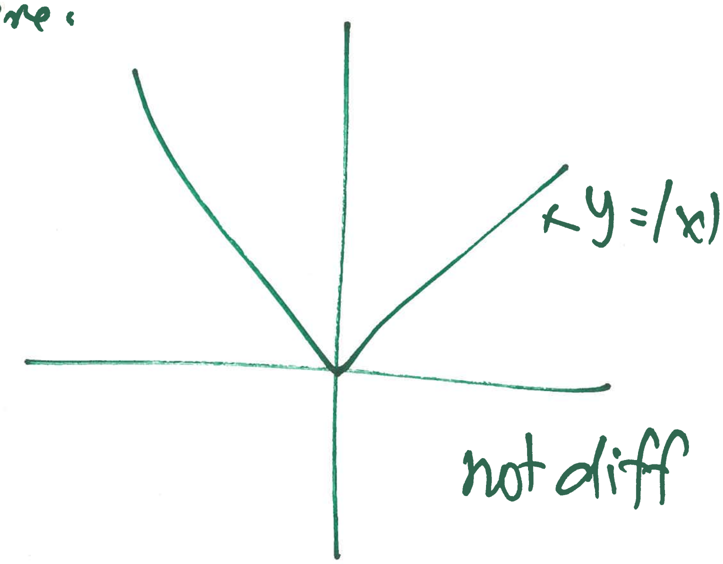
compare:



not diff at $x = 0$

3. The tangent line

(6) ⋆⋆ (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

$f(x) = x^{\frac{1}{2}}$, so $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, so $f'(4) = \frac{1}{4}$

so line is $y = 2 + \frac{1}{4}(x-4)$

$= \frac{1}{4}x + 1$

(7) ⋆⋆(Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

line passes through $(1, 6)$, has slope 4.

(8) $\star\star\star$ Find the lines of slope 3 tangent the curve $y = x^3 + 4x^2 - 8x + 3$.

Let a be the x-coord of a point of tangency, then $y'(a) = 3$. But $y' = 3x^2 + 8x - 8$

so $3a^2 + 8a - 8 = 3$ so $3a^2 + 8a - 11 = 0$

so $a = \dfrac{-8 \pm \sqrt{64 + 132}}{6} = \dfrac{-8 \pm 14}{6} = 1, \dfrac{-11}{3}$

(9) $\star\star\star$ The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

4. Linear approximation

Definition. $f(a+h) \approx f(a) + f'(a)h$

(10) Estimate

(a) $\star$ $\sqrt{1.2}$

Let $f(x) = \sqrt{x}$, $f(1) = 1$, $f'(1) = \left[\frac{1}{2} x^{-\frac{1}{2}}\right]_{x=1} = \frac{1}{2}$

so $f(x) \approx 1 + \frac{1}{2}(x-1)$

so $f(1.2) \approx 1 + \frac{1}{2}(.2) = 1.1$

(b) $\star$ (Final, 2015) $\sqrt{8}$