1A)

4. Computing derivatives (27/9/2023)

Goals.

(1) Combining linear approximations
(2) Linearity of the derivative
(3) The product and quotient rules

Last Time. Derivative

Def: The **derivative** of f at a is the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

if the limit exists. Write $f'(a)$, $\frac{df}{dx}(a)$, $\left.\frac{df}{dx}\right|_{x=a}$, ..

Equivalently, if $f(a+h)-f(a) \sim mh$ as $h\to 0$
m is the derivative

$\Rightarrow$ linear approx $f(a+h) \approx f(a) + f'(a)h$

Also got **derivative function** $a \mapsto f'(a)$
if f is differentiable on an interval.

WS 1 (a), (b), (c)

Math 100A – WORKSHEET 4
COMPUTING DERIVATIVES

1. Review of the derivative

(1) Expand $f(x+h)$ to linear order in h for the following functions and read the derivative off:

(a) ⋆ $f(x) = bx$

$f(x+h) = b(x+h) = bx + bh$, so $f'(x) = b$

(b) ⋆ $g(x) = ax^2$

$g(x+h) = a(x+h)^2 = ax^2 + 2ahx + ah^2 \approx ax^2 + 2ax \cdot h$

to 1st order in h

so $g'(x) = 2ax$

(c) ⋆ $h(x) = ax^2 + bx$.

Can repeat: $h(x+h) = a(x+h)^2 + b(x+h) = (ax^2 + bx) + (2ax+b) \cdot h + ah^2$

$\approx (ax^2 + bx) + (2ax+b)h$

so $h'(x) = 2ax + b$

Or: $h = f + g \Rightarrow h(x+h) = f(x+h) + g(x+h) \approx f(x) + g(x) + 2axh + bh$ to 1st order

Date: 27/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Conclusion: Since ~~add~~ the sum of terms dominated by h is dominated by h, we can add linear approximations

$$\Rightarrow \text{If } f(x+h) \approx f(x) + f'(a)h$$
$$g(x+h) \approx g(x) + g'(a)h$$
$$\Downarrow$$
$$(f+g)(x+h) = f(x) + g(x) + f'(a)h + g'(a)h$$
$$= (f+g)(x) + (f'(a) + g'(a)) \cdot h$$

So $\boxed{(f+g)' = f' + g'}$

WS 1(e)

Fact: If we are working to linear/1^{st} order, can replace any part of $f(x+h)$ by its linear approximation.

Like above, can use this to show $\boxed{(\alpha f + \beta g)' = \alpha f' + \beta g'}$ ("linearity of the derivative")

(d) ⋆⋆ $i(x) = \frac{1}{b+x}$

(e) ⋆⋆⋆ $j(x) = 4x^4 + 5x$ (hint: use the known linear approximation to $2x^2$)

$j(x) = (2x^2)^2 + 5x$ so $j(x+h) = (2(x+h)^2)^2 + 5(x+h)$

$\approx (2x^2+4xh)^2 + (5x+5h)$

neglected $16x^2h^2$ ~~which was~~ $\rightarrow \approx 4x^4 + 16x^3 h + 5x + 5h$

$\approx (4x^4+5x) + (16x^3+5)h$

so $j'(x) = 16x^3 + 5$

Message: "If we see something that looks like $f(x+h)$, can try replacing it with

$f(x) + f'(x)h$".

WS 2: Apply diff rules (recall $(x^n)' = nx^{n-1}$
$(e^x)' = e^x$)

~~Example~~ (linear approx = find line tangent to f at $(x, f(x))$, evaluate line at $x+h$)

Example: Say $f(x+h) \approx f(x) + f'(x)h$

$g(x+h) \approx g(x) + g'(x)h$

Then $f(x+h)\,g(x+h) \approx \big(f(x) + f'(x)h\big)\big(g(x) + g'(x)h\big)$

$\approx f(x)g(x) + f(x)g'(x)h + f'(x)h \cdot g(x) + f'(x)h\,g'(x)h$

$= f(x)g(x) + \big(f(x)g'(x) + f'(x)g(x)\big)h + f'(x)g'(x)h^2$

$\approx f(x)g(x) + \big(f(x)g'(x) + f'(x)g(x)\big)h$

so $\boxed{(fg)'(x) = f(x)g'(x) + f'(x)g(x)}$ $\boxed{(fg)' = f'g + f \cdot g'}$

("product rule")

Similarly, using $\frac{1}{x+h} \approx \frac{1}{x} - \frac{1}{x^2}h$ get

$\boxed{\left(\frac{1}{g}\right)' = -\frac{1}{g^2}g'}$, $\boxed{\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{f}{g^2}g' = \frac{f'g - fg'}{g^2}}$

("quotient rule")

2. Arithmetic of derivatives

(2) Differentiate

(a) $\star$ $f(x) = 6x^\pi + 2x^e - x^{7/2}$ is a sum, apply sum rule

$f'(x) = 6\pi \cdot x^{\pi-1} + 2e \cdot x^{e-1} - \frac{7}{2} x^{5/2}$

(b) $\star$ (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

(product of x^2, e^x)

$$\frac{d}{dx}(x^2 e^x) = \left(\frac{d}{dx} x^2\right) e^x + x^2 \left(\frac{d}{dx} e^x\right) = 2x e^x + x^2 e^x$$

(c) ⋆ (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$h'(x) = \frac{(x^2+3)'\cdot(2x-1) - (x^2+3)(2x-1)'}{(2x-1)^2} = \frac{(2x)(2x-1)-(x^2+3)(2)}{(2x-1)^2}$$

$$= \frac{2x^2-2x-6}{(2x-1)^2}$$

(d) ⋆ $\frac{x^2+A}{\sqrt{x}}$ can use quotient rule

$$\left(\frac{x^2+A}{\sqrt{x}}\right)' = \frac{2x\cdot\sqrt{x} - (x^2+A)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= 2\sqrt{x} - \frac{1}{2}x^{\frac{1}{2}} - \frac{A}{2}x^{-3/2} \qquad \frac{3}{2}\sqrt{x} - \frac{A}{\ldots}$$

$$\left(\frac{x^2+A}{\sqrt{x}}\right)' = \left(x^{3/2} + Ax^{-\frac{1}{2}}\right)' = \frac{3}{2}x^{\frac{1}{2}} - \frac{A}{2}x^{-3/2}$$

(3) ⋆ Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

$$f'(x) = \frac{(\sqrt{x}+A) - x\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2} \quad \text{so } f'(4) = \frac{1+A}{(2+A)^2}$$

$$\text{so } \frac{1+A}{(2+A)^2} = \frac{3}{16} \quad \text{so } 16(1+A) = 3(2+A)^2$$

$$\text{so } \boxed{3A^2 - 4A - 4 = 0}$$

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) $\star$ What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.

$f(1+h) \approx 1 + 3h$, $g(1+h) \approx 2 + 4h$

so $(fg)(1+h) \approx (1+3h)(2+4h) \approx 2 + 10h + 12h^2$

$\approx 2 + 10h$

(or: $(fg)(1) = 2$, $(fg)'(1) = f'(1)\,g(1) + f(1)\,g'(1) = 10$

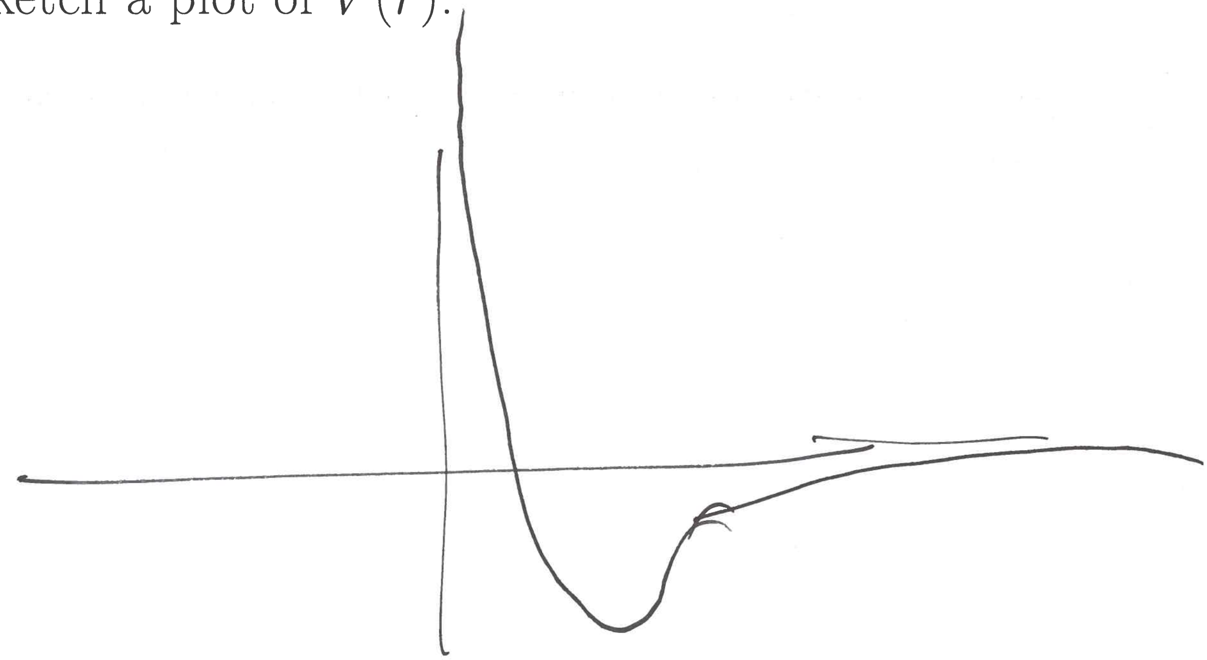
(b) $\star$ Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

(6) The *Lennart–Jones potential* $V(r) = \epsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right)$ models the electrostatic potential energy of a diatomic molecule. Here $r > 0$ is the distance between the atoms and $\epsilon, R > 0$ are constants.

(a) $\star$ What are the asymptotics of $V(r)$ as $r \to 0$ and as $r \to \infty$?

As $r \to 0$, $V(r) \sim \varepsilon\left(\frac{R}{r}\right)^{12}$, as $r \to \infty$ $V(r) \sim -2\varepsilon\left(\frac{R}{r}\right.$

(b) Sketch a plot of $V(r)$.



(c) Find the derivative $\frac{dV}{dr}(r) =$

$$V'(r) = -\varepsilon\, 12 \frac{R^{12}}{r^{13}} + 12\varepsilon \frac{R^6}{r^7} = \frac{12\varepsilon}{r}\left(\left(\frac{R}{r}\right)^6 - \left(\frac{R}{r}\right)^{12}\right)$$

(d) Where is $V(r)$ increasing? decreasing? Find its minimum location and value.