1A1

5. The Chain rule (4/10/2023)

Goals.

(1) The Chain Rule
(2) Logarithmic differentiation

Last Time. Arithmetic of derivatives:

(1) **Linearity**: $(af+bg)' = af' + bg'$ (a, b constants)

(2) **Product rule**: $(fg)' = f'g + fg'$

(3) **Quotient rule**: $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$; $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Can be used ~~for~~ on intervals (to get the derivative as a function)

or pointwise.

Previously: linear approximation

$f(a+h) \approx f(a) + f'(a)h$

Math 100A – WORKSHEET 5
THE CHAIN RULE

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.

(a) ⋆Expand $\sin(y+h)$ to linear order in h. Write down the linear approximation to $\sin y$ about $y = a$.

Since $(\sin y)' = \cos y$,

$\sin(y+h) \approx \sin y + \cos y \, h$

(Aside: have $\sin(y+h) = \sin y \cosh + \cos y \sin h$
can use $\cos h \approx 1$ to 1^{st} order, $\sin h \approx h$ to 1^{st} order)

(b) ⋆⋆Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

thus

$F(x+h) = \sin(3(x+h)) = \sin(3x+3h) = \sin(y+3h)$

$y = 3x$

$\approx \sin y + (\cos y)\cdot 3h$

$\approx \sin(3x) + 3(\cos 3x)\cdot h$

Thus $\frac{d}{dx}(\sin(3x)) = 3\cdot\cos(3x)$

Date: 4/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

~~So~~ The same argument shows:

$$\boxed{\frac{d}{dx} f(ax) = a \cdot \frac{df}{dx}(ax)}$$

In general, say $F(x) = g(f(x))$
for functions f, g. Suppose f diff. at x,
~~There have~~ g diff. at $y = f(x)$

Then $f(x+h) \simeq f(x) + f'(x)h$ are linear approx
$g(y+k) \simeq g(y) + g'(y)k$

lin. approx ~~to~~ f

So $F(x+h) = g(f(x+h)) = g(\underbrace{f(x)}_{y} + \underbrace{f'(x)h + \text{error}}_{k})$

$\simeq g(f(x)) + g'(y) \cdot (f'(x)h + \text{error})$

lin. approx. to g

$\simeq g(f(x)) + \underbrace{g'(y) f'(x)}_{F'(x)} h$

or $$\boxed{F'(x) = g'(f(x)) \cdot f'(x)}$$

Chain rule

Avoid error: $\frac{d}{dx} f(g(x)) \neq f'(x) g'(x)$

Another way to phrase this:

$$\frac{d(g(f(x)))}{dx} = \frac{dg}{df} \cdot \frac{df}{dx}$$

E.g. if $y = 3x$

$$\frac{d(\sin(3x))}{dx} = \frac{d(\sin(3x))}{d(3x)} \cdot \frac{d(3x)}{d(x)}$$

$$= \cos(3x) \cdot 3.$$

(2) Write each function as a composition and differentiate

(a) ⋆ e^{3x}

let $f(y) = e^y$, $g(x) = 3x$

Then $e^{3x} = f(g(x))$

So $\frac{d}{dx}(e^{3x}) = f'(g(x)) \cdot g'(x) = e^y \cdot 3 = e^{3x} \cdot 3$

Or $\frac{df}{dy} = e^y$, $\frac{dy}{dx} = 3$, so $\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dg}{dx} = e^y \cdot 3$
$= 3e^{3x}$.

(b) ⋆ $\sqrt{2x+1}$

Here $\sqrt{2x+1} = \sqrt{y}$ where $y = 2x+1$

So $\frac{d(\sqrt{2x+1})}{dx} = \frac{d(\sqrt{y})}{dy} \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{y}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

(c) (Final, 2015) $\star$ $\sin(x^2)$

Let $\theta = x^2$, ~~θ~~ in which terms we have $\sin\theta$.

So $\frac{d(\sin\theta)}{dx} = \frac{d(\sin\theta)}{d\theta} \cdot \frac{d\theta}{dx} = \cos\theta \cdot (2x)$

$= 2x\cos(x^2)$

(d) $\star$ $(7x + \cos x)^n$.

This is $f(g(x))$ where $f(y) = y^n$, $g(x) = 7x + \cos x$

so $f(g(x))' = f'(g(x)) \cdot g'(x) = n(7x + \cos x)^{n-1} \cdot (7 - \sin x)$

(3) (Final, 2012) $\star\star$ Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

By chain rule

$f'(x) = g'(2\sin x) \cdot (2\cos x)$

So $f'\left(\frac{\pi}{4}\right) = g'\left(2 \cdot \sin\frac{\pi}{4}\right) \cdot 2\cos\frac{\pi}{4} = g'(\sqrt{2}) \cdot \sqrt{2} = 2$.

(4) Differentiate

(a) $\star$ $7x + \cos(x^n)$

$$\left(7x+\cos(x^n)\right)' \underset{\uparrow}{=} 7 + \left(\cos(x^n)\right)' \underset{\uparrow}{=} 7 - \sin(x^n)\cdot nx^{n-1}$$

sum rule chain rule

(b) $\star$ $e^{\sqrt{\cos x}}$

$$\frac{d}{dx} e^{\sqrt{\cos x}} = e^{\sqrt{\cos x}} \cdot \frac{1}{2}(\cos x)^{-\frac{1}{2}} (-\sin x) = -\frac{1}{2} e^{\sqrt{\cos x}} \cdot \frac{\sin x}{\sqrt{\cos x}}$$

or $$= \frac{d(e^{\sqrt{\cos x}})}{d(\sqrt{\cos x})} \cdot \frac{d(\sqrt{\cos x})}{d(\cos x)} \cdot \frac{d(\cos x)}{dx} = \ldots$$

(c) $\star$ (Final 2012) $e^{(\sin x)^2}$

$$\frac{d}{dx} e^{(\sin x)^2} = e^{(\sin x)^2} \cdot \underbrace{2\sin x \cdot \cos x}_{\frac{d(\sin x)^2}{dx}}$$

power law rule

chain rule

chain rule

(5) $\star\star$ Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Differentiating both sides we get

$$f'(g(x)) \cdot g'(x) = 3x^2$$

so $f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$ so $\boxed{g'(4) = \frac{48}{5}}$

Logarithms

Main point: $\log(xy) = \log x + \log y$

$\Rightarrow \log(x^y) = y\log x$

(converts products to sums
powers to products)

We write $\log x$ for logarithms to the **natural** base.

Know: $y = \log x$, then $x = e^y$ take $\frac{d}{dx}$

$$1 = \underbrace{e^y \cdot \frac{dy}{dx}}_{\text{chain rule}} = e^y \cdot \frac{d(\log x)}{dx}.$$

So $\boxed{\dfrac{d(\log x)}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}.}$

2. Logarithmic differentiation

(6) $\star$ $\log\left(e^{10}\right) = 10$ $\qquad$ $\log(2^{100}) = 100 \log 2$

(7) $\star$ Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} = \frac{1}{ax} \cdot a = \frac{1}{x}$

or: $\log(ax) = \log a + \log x$

$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) = \frac{2t+3}{t^2+3t}$

(b) $\star$ $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$

$= 2x \log(1+x^2) + x^2 \cdot \frac{2x}{1+x^2}$

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$

$= -\left(\log(2+\sin r)\right)^{-2} \cdot \frac{\cos r}{2+\sin r}$

(aside: check that $\frac{d \log|x|}{dx} = \frac{1}{x}$)

If f any function, $(\log f)' = \frac{1}{f} \cdot f'$

so $\boxed{f' = f \cdot (\log f)'}$

Logarithmic diff. rule

(8) ⋆⋆ (Logarithmic differentiation) differentiate

$$y = (x^2+1) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}.$$

$$\log y = \log(1+x^2) + \log(\sin x) - \frac{1}{2}\log(x^3+3) + \cos x$$

$$\text{So } \frac{y'}{y} = \frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$$

$$\text{So } \frac{dy}{dx} = \underbrace{\left((x^2+1)\sin x \frac{1}{\sqrt{x^3+3}} e^{\cos x}\right)}_{\text{can't have } y \Rightarrow \text{ want fcn of } x.}\left(\frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x\right)$$

(9) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) ⋆ x^n

$$(x^n)' = x^n \cdot (\log(x^n))' = x^n (n \log x)' = x^n \cdot \left(\frac{n}{x}\right) = n x^{n-1}.$$

(b) $\star$ x^x

$$(x^x)' = x^x \cdot (\log(x^x))' = x^x \cdot (x \cdot \log x)' = x^x\left(\log x + \frac{x}{x}\right)$$

$$= x^x(\log x + 1).$$

(c) $\star\star$ $(\log x)^{\cos x}$

$$\log\left((\log x)^{\cos x}\right) = \cos x \log\log x$$

so $$\frac{d}{dx}\left((\log x)^{\cos x}\right) = (\log x)^{\cos x} \cdot \frac{d}{dx}(\cos x \log\log x)$$

$$= (\log x)^{\cos x} \cdot \left(-\sin x \cdot \log\log x + \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x}\right)$$

(d) (Final, 2014) $\star$ Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x only.