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9. Optimization (1/11/2023)

Goals.

(1) Review: calculus and the shape of the graph
(2) Optimization of functions
(3) Problem solving: optimization problems

Last Time. Taylor expansion

(1) To approximate f near x=a create polynomial $T_n(x)$ matches first n derivatives of f at a:

set $\boxed{c_k = \frac{f^{(k)}(a)}{k!}}$ then $T_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)$

(2) Can combine approximations:

(a) If T_f, T_g approximate f to n'th order at a, then $\alpha T_f + \beta T_g$, $T_f T_g$ approximate $\alpha f + \beta g$, fg respectively to n'th order at $x=a$.

(b) If $T_g(x)$ approximates g to n'th order at $x=a$,
$T_f(u)$ " f " " " " $u=b=g(a)$
then $T_f(T_g(x))$ approximates $f\circ g$ to n'th order about $x=a$

(c) Know: $e^u \approx 1+u+\frac{u^2}{2!}+u^3/3!+\dots$ | $\frac{1}{1-u} = 1+u+u^2+u^3+\dots$

Shape of graph

Given function f on $[a,b]$
can tell if f is increasing/decreasing using f'.
If $f'(x) \neq 0$, x <u>not</u> local max or local min.

$\Rightarrow$ local max/min only occur where $\begin{cases} f'(x) = 0 & \leftarrow \text{critical pt} \\ \text{or } f'(x) \text{ undef} & \leftarrow \text{singular pt} \end{cases}$

<u>Theorem:</u> Let f be continuous on a closed interval $[a,b]$
Then f <u>has</u> a global max & global min on $[a,b]$,
attained at either (1) critical pt; or
(2) singular pt; or
(3) end point of $[a,b]$

Don't need "tests" for max/min: just take largest value

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

f is continuous on $[-5,5]$ (polynomial).

$f'(x) = 4x^3 - 8x = 4x(x^2-2) = 4x(x-\sqrt{2})(x+\sqrt{2})$

Critical pts where $x=0$, $x=\pm\sqrt{2}$

$f(\pm 5) = 625 - 100 + 4 = 529$

$f(\pm\sqrt{2}) = 0$

$f(0) = 4$

$\Rightarrow$ max value is 529, attained at ± 5; min value is 0, attained at $\pm\sqrt{2}$.

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

Now only critical point is at $x=0$, and $f(\pm 1) = 1$

So min is 1, at $x=\pm 1$

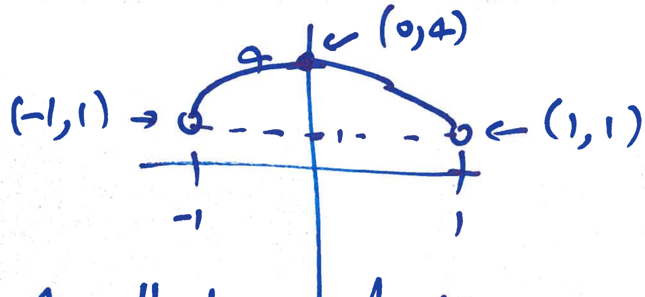
max is 4, at $x=0$

Date: 1/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Lesson: domain matters. $\pm\sqrt{2}$ are zeroes of $4x^3-8x$, but outside $[-1,1]$

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

Graph:



Max is 4, attained at $x=0$

But **no minimum value** ("infimum" is 1, but it is never attained)

Lesson: on open interval, no promise of min or max

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

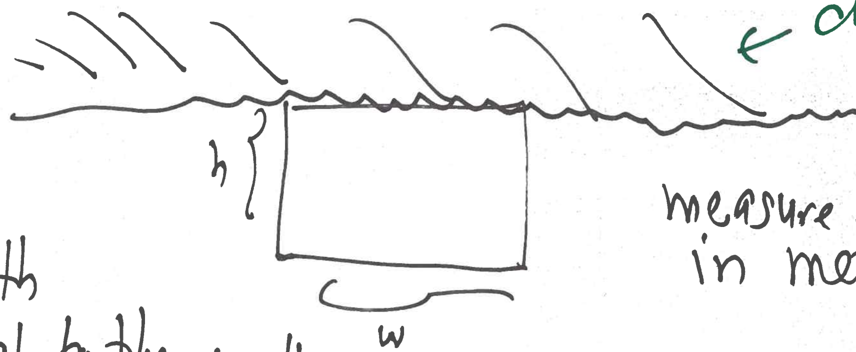
At $\pm\infty$, $f(x) \sim x^4 \to \infty$ so f not bounded above, no max

Minimum must be at critical pt

(e.g. $f(100) \approx 10^4$, $f(-100) \approx 10^4$ on $[-100, 100]$ min is a crit pt, outside f too big to give min)

$f(0) = 4$, $f(\pm\sqrt{2}) = 0$ so min is 0, attained at $\pm\sqrt{2}$.

(5) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?



← diagram

measure lengths in metres

chosen names

let w be the length of the side parallel to the wall, h the length of the side perpendicular to the wall
let A be the area of the rectangle

Then $A = h \cdot w$, and $2h + w = 100$ so $w = 100 - 2h$

and $A = h(100-2h)$

finding & enforcing relation

↖ objective fcn

makes sense if $h \geq 0$, and $h \leq 50$ (cause at most 100m of fencing)

so want max $A(h)$ for $0 \leq h \leq 50$

include **degenerate** possibilities $h=0$, $h=50$

A cts, diff on $[0,50]$

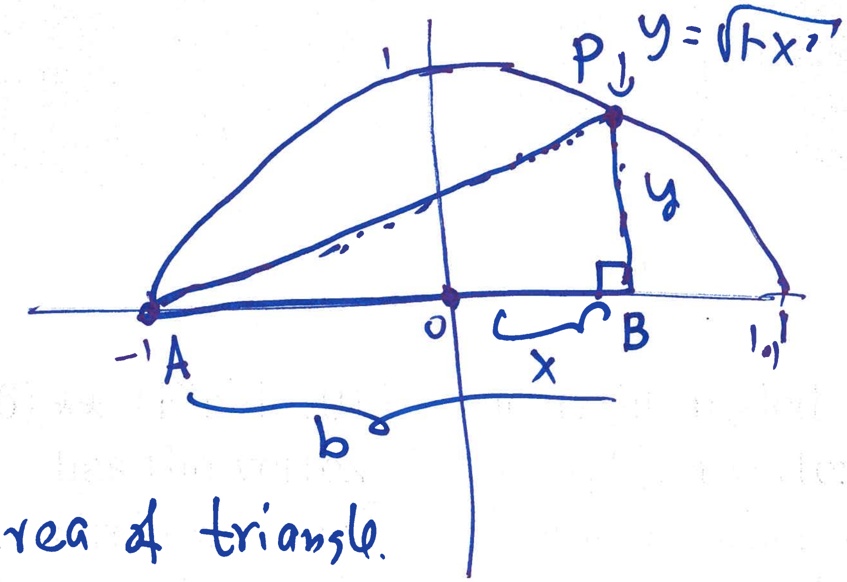
$\frac{dA}{dh} = 100 - 2h - 2h = 100 - 4h$, vanishes at $h = 25$m.

$A(0) = A(50) = 0$, $A(25) = 25 \cdot 50 = 1250\,m^2$

Calculus

so the largest area possible is $1250\,m^2$. endgame

(6) ⋆⋆ (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semi-circle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?



say $P = (x, y)$

b = base of triangle

$P = (x, y)$

$B = (x, 0)$

$-1 \le x \le 1$

S = area of triangle.

then $b = 1 + x$

$y = \sqrt{1 - x^2}$

and $S = \frac{1}{2} b \cdot y = \frac{1}{2}(1+x)\sqrt{1-x^2}$

want max of S on $[-1, 1]$: $S^2 = \frac{(1+x)^2(1-x^2)}{4}$

$= \frac{(1+x)^3(1-x)}{4}$

so $2SS' = \frac{1}{4}\left[3(1+x)^2(1-x) - (1+x)^3\right]$

so $8SS' = (1+x)^2\left(3(1-x) - (1+x)\right]$

$= (1+x^2)(2-4x)$

So $\quad S' = \dfrac{2(1+x)^2(1-2x)}{4(1+x)\sqrt{1-x^2}}$

(undef at $x = \pm 1$)

critical pt where $x = \frac{1}{2}$, and

$S(-1) = S(1) = 0$, $\quad S\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{8}$

so largest area is $\frac{3\sqrt{3}}{8}$

(attained for $P = \left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right)$)

(7) A ferry operator is trying to optimize profits. Before each ferry trip workers spend some time loading cars after which the trip takes 1 hour. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour and the fuel for the trip costs \$250. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.

independent variable is t

revenue: $50 \cdot N(t) = 5{,}000 \frac{t}{t+1}$

Costs : $250 + 500t$

Profit: $P(t) = 5{,}000 \frac{t}{t+1} - 500t - 250$

want max $P(t)$ for $0 \le t < \infty$.

$P(0) = -250$, as $t \to \infty$, $P(t) \to -500t$

$P(1) = 2{,}500 - 500 - 250 = 1{,}750 > 0$

so max is somewhere in the middle $\Rightarrow$ at crit pt

$P'(t) = \frac{5{,}000}{(t+1)^2} - 500 \Rightarrow$ crit pt at $(t+1)^2 = 10$

so $t = \sqrt{10} - 1$ hours

is best time