local min if for all x close to x_0, $f(x) \geq f(x_0)$

10. Differential Equations (8/11/2023)

Goals.

(1) Differential equations
  (a) What is a differential equation?
  (b) What isa solution to a differential equation?
  (c) Plugging ansatze into equations

(max) $(f(x) \leq f(x_0))$

global min on $[a,b]$ if for all $x \in [a,b]$ $f(x) \geq f(x_0)$
(max) $(f(x) \leq f(x_0))$

(endpoints can be local min/max)

Theorem: f cts on closed interval $[a,b]$, then:
(1) global max & min exist.
(2) any local extremum is at one of critical pts, singular pts, endpoints.

Problem-solving: Especially getting started by giving names.

Today: **Equations**

Equations

Usually $A=B$ is an assertion: a claim that A, B are equal (could be true or false)

E.g. $5+7=6+6$. (true)

$$\frac{d(x^2)}{dx} = x \qquad \text{(false)}$$

Have same setup $A=B$, but now A, B depend on variables. No assertion, but question: for which values of variables, is it true that $A=B$?

Typical assertion: "$y=3$ is a solution"
check by "plugging in" see if we get true equality.

Differential equations

unknowns are functions instead of numbers (equation involves derivatives)

Example: (Newton): $F=ma$, $m\cdot\frac{d^2x}{dt} = F(x,v,t$

$y=2$ is a solution: $2^2 \stackrel{?}{=} 4$ | $2^2 = 4 \neq 6 = 3\cdot 2$

in fact ± 2 are all solutions: | all solutions: $0, 3$

Math 100A – WORKSHEET 10
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad\qquad y^2 = 3y$$

$y^2 = 4 \iff (y-2)(y+2) = 0$ | $y^2 = 3y \iff (y-3)\cdot y = 0$

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad\qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

$\frac{d(x^2)}{dx} = 2x \neq x^2$ ✗ | $\left(\frac{d(x^2)}{dx}\right)^2 = 4x^2 = 4\cdot x^2$ ✓

$\frac{d(e^x)}{dx} = e^x$ ✓ | $\left(\frac{d(e^x)}{dx}\right)^2 = e^{2x} \neq 4e^x$

Date: 8/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(3) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

LHS: $2t + t^2 - 1$ $\neq$

RHS: t^2

incomplete attempt:

plugging in:

$2t + 2 + t^2 - 1 \overset{?}{=}$

$= t^2 + 2t + 1 = (t+1)^2$ ✓

Aside: $\frac{dz}{dt} - z = 1 - t^2$ is linear in z

C = Constant:

$\leftarrow y = C\cdot e^{1.04t}$

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

try: $y = 1.04\cdot e^{t}$ $\quad \frac{dy}{dt} = 1.04\cdot e^{t} = y \neq 1.04y$

$e^{1.04\cdot t}$

the statement $y(0)=100$ is an equation for C!

$C\cdot e^{0} = 100$ so $C=100$, $y(t) = 100\cdot e^{1.04t}$

particular solution

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

$y' = ay$, $z' = bz$

$$w' = \left(\frac{y}{z}\right)' = \frac{y'z - yz'}{z^2} \stackrel{!}{=} \frac{ayz - y\cdot bz}{z^2}$$

$$= a\frac{y}{z} - b\frac{y}{z} = (a-b)\cdot w$$

General solution: $= A\sin(2t) + B\cos(2t)$)

family of functions = Ansatz

2. Solutions by massaging and ansatze

(7) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

This y is a solution when $-\omega^2 \sin(\omega t) + 4\sin(\omega t) = 0$

$\iff (4-\omega^2)\sin(\omega t) = 0$

So a solution when $\omega = \pm 2$ $\omega = 0$

Can select a particular solution using an initial condition or other constraints

Summary

(1) Diff eqn := equality of functions involving:

(1) unknown functions

(2) derivatives of the unknowns

(2) A function is a solution if plugging it into equation gives true equality

(4) General solution to $y' = ry$ is $y(t) = C \cdot e^{rt}$ where C is an arbitrary constant

(b) What is the general solution for $u(t)$?

a, b constants

(9) Consider the equation $\frac{dy}{dt} = a(y-b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by u?

$\frac{dy}{dt} = y' = a(y-b) = au$

So $u(t) = C' \cdot e^{at}$

C' arbitrary.

(c) What is the general solution for $y(t)$?

$y(t) = u(t) + b = b + C' \cdot e^{at}$

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?

e^{at} decays exponentially, so $y(t) \xrightarrow[t\to\infty]{} b$ exponentially.

(aside: $C=0$ get solution $\boxed{y(t) \equiv b}$)

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?

$\boxed{u' = -ku}$

(10) Example: *Newton's law of cooling.* Suppose we place an object of temperature $T(0)$ in an environment of temperature T_{env}. It turns out that a good model for the temperature $T(t)$ of the object at time t is

$$\frac{dT}{dt} = -k\,(T - T_{\text{env}})$$

where $k > 0$ is a positive constant.

(a) Suppose $T(t) > T_{\text{env}}$. Is $T'(t)$ positive or negathen have $u(0) = 32.5 - 20$ } two equations for C, k

(b) A body is found at 1:30am and its temperature is measured to be 32.5°C. At 2:30am its temperature is found to be 30.3°C. The temperature of the room in which the body was found is measured to be 20°C and we have no reason to believe the ambient temperature has changed. What was the time of death?

$u(1) = 30.3 - 20$ $\boxed{-kt}$ $u(t) = T_{body}(t) - 20$

solve for C, k.

<u>Q</u>: find t s.t. $u(t) = 37°C$

$u(0) = C = 12.5$

$u(1) = Ce^{-k} = 10.3 \Rightarrow e^{-k} = \frac{10.3}{12.5} \Rightarrow k = \log \frac{12.5}{10.3}$

sanity check $k > 0$

Need t s.t. $12.5 \cdot e^{-\log \frac{12.5}{10.3} t} = 37° - 20°$

$\Rightarrow t = -\frac{\log(17/12.5)}{\log(12.5/10.3)}$ hours before 1:30

sanity check: $t < 0$

$\approx y_k + F(y_k)\, h$

Numerics

Idea: linear approximation

DE $\quad y' = F(y) \qquad ; \quad y' = e^y$

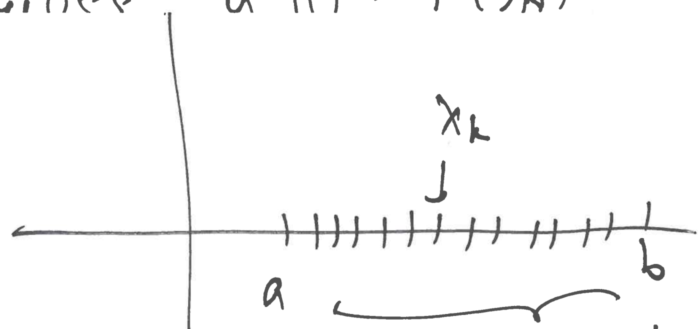
$\qquad\qquad\qquad\qquad\qquad\quad y' = 5y$

$\qquad\qquad\qquad\qquad\qquad\quad \vdots$

if have guess $y(t) \approx y_k$

$\Downarrow$

Guess $\; y'(t) \approx F(y_k)$



n intervals of length $h = \frac{b-a}{n}$

y_k = value at x_k $\quad$ start at y_0

Guess: $y_{k+1} \approx y_k + F(y_k) \cdot h$