1A)

12. Multivariable calculus (24/11/2023)

Goals.

(1) 3d space: coordinates and graphs
(2) Partial derivatives

Last Time. **Numerical methods**

① Euler scheme: method for numerically solving ODE.

Input: ① ODE $y' = f(y;t)$
② initial condition (t_0, y_0)
③ Endpoint b.

solve ODE on $[y_0, b]$

Algorithm: choose stepsize $h = \frac{b-y_0}{n}$, n = number of subintervals

let $t_k = t_0 + kh$:

$t_1 = t_0 + h$, $t_2 = t_0 + 2h = t_1 + h$, ..., $t_n = b$.

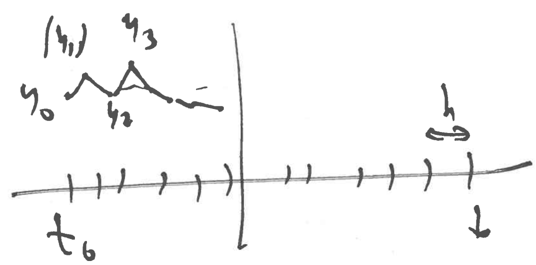
Set y_0 as given, calculate $y_1 = \underbrace{y_0 + f(y_0;t_0)h}_{\text{linear approx}}$ plug (t_0, y_0) into ODE

$y_2 = y_1 + f(y_1;t_1)h$

slope from new point

$\vdots$

$$\boxed{y_{k+1} = y_k + f(y_k, t_k)h}$$



② Newton's Method

points where $f(x)=0$

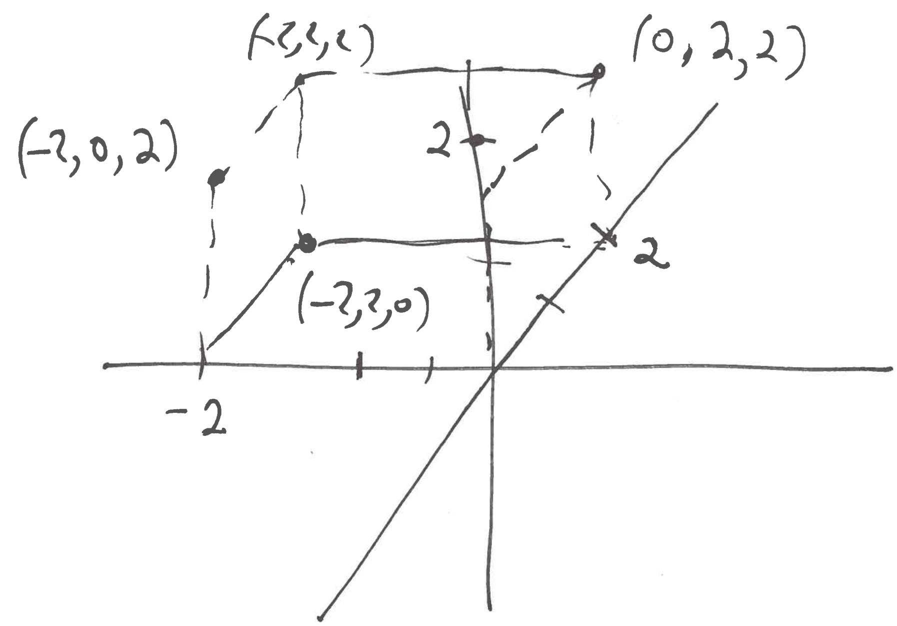
Method for finding zeroes of functions

Inputs function f, point x_0 close to a zero.

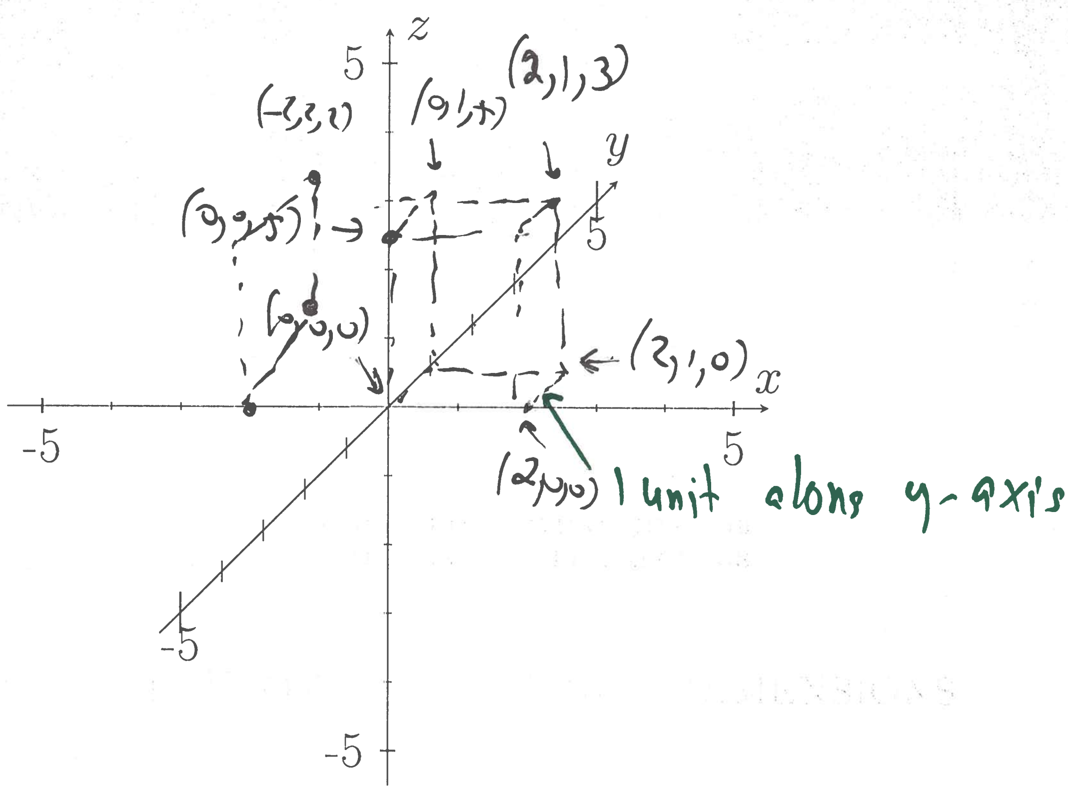
Algorithm: Given guess x_k do linear approx to f about x_k, find point x_{k+1} where linear approx is 0:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

The Graph of $z=f(x,y)$ is a surface in 3d space (like graph $y=f(x)$ is a curve in 2d space)



1. Plotting in three dimensions



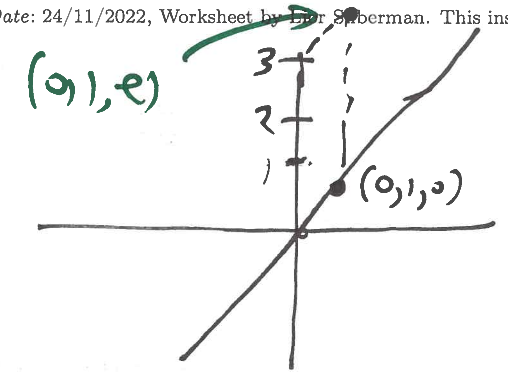
(1) $\star$ Plot the points $(2,1,3)$, $(-2,2,2)$ on the axes provided.

$f(0,-1) = e^{0^2+(-1)^2} = e$, $f(1,2) = e^5$

(2) Let $f(x,y) = e^{x^2+y^2}$.

(a) $\star$ What are $f(0,-1)$? $f(1,2)$? Plot the point $(0,1,f(0,1))$ on the axes provided.

$f(0,1) = e$



Remarks

(1) Have 3d plotting tools, e.g. Desmos 3D.

(2) Important intuition: surface of the Earth
= topographical maps

(3) useful visualization tool: level curves.

Example: level curves of $f(x,y) = e^{x^2+y^2}$
are circles: $e^{x^2+y^2} = z \iff x^2+y^2 = \log z$

~~Example domain~~

(b) $\star$ What is the *domain* of f (that is: for what (x, y) values does f make sense?

Domain is : whole plane

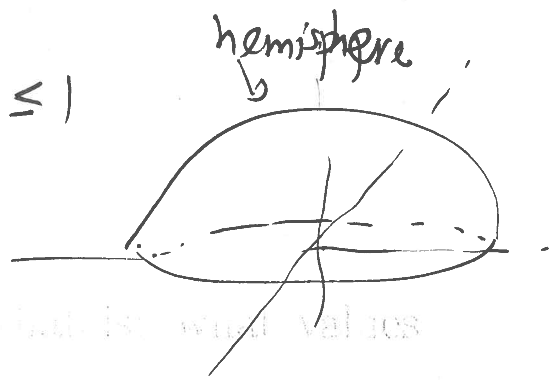
(c) $\star$ What is the *range* of f (that is: what values does it take)?

range of x^2+y^2 is $[0,\infty)$ so range of $e^{x^2+y^2}$ is $[1,\infty)$:
$e^{x^2+y^2} \geq e^0 = 1$

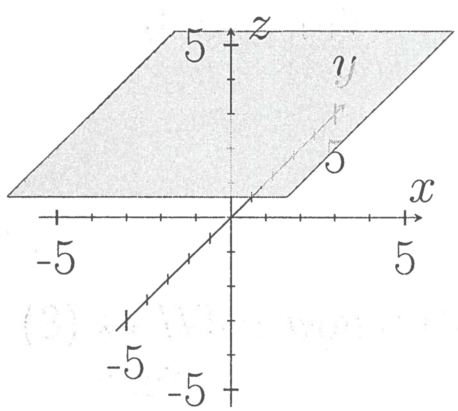
(3) $\star\star$ What would the graph of $z = \sqrt{1-x^2-y^2}$ look like?

domain is $1-x^2-y^2 \geqslant 0$, so $x^2+y^2 \leq 1$

$z = \sqrt{1-x^2+y^2} \Rightarrow \begin{cases} z^2+x^2+y^2=1 \\ z \geqslant 0 \end{cases}$



(4) $\star$ Which plane is this?



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

plane parallel to xy-plane through $(0,0,3)$

2. Partial derivatives

(5)(a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

$$\frac{df}{dx} = 4x$$

(b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

$$\frac{df}{dx} = 4x$$

(c) $\star$ Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

$$\frac{\partial f}{\partial x} = 4x$$

"partial derivative of f wrt x with y constant"

\partial = ∂

(d) $\star$ What is $\frac{\partial f}{\partial y}$?

$$\frac{\partial f}{\partial y} = -2y$$

(6) Find the partial derivatives with respect to both x, y of

(a) $\star$ $g(x, y) = 3y^2 \sin(x+3)$

$$\frac{\partial g}{\partial x} = 3y^2 \cos(x+3)$$

$$\frac{\partial g}{\partial y} = 3 \cdot 2y \cdot \sin(x+3) = 6y \sin(x+3)$$

(b) $\star$ $h(x, y) = ye^{Axy} + B$

$$\frac{\partial h}{\partial x} = ye^{Axy} \cdot (Ay) = Ay^2 e^{Axy}$$

$$\frac{\partial h}{\partial y} \underset{\uparrow}{=} e^{Axy} + y \frac{\partial}{\partial y} e^{Axy} = e^{Axy} + y e^{Axy} Ax$$

pdt rule

$$= (1 + Axy) e^{Axy}.$$

(7) The the gravitational *potential* due to a point mass M (equivalently the electrical potential due to a point charge M) is given by the formula $U(x, y, z) = \frac{GM}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$. Here G is the universal gravitational constant (equivalently G is the Coulomb constant).

(a) $\star$ The x-component of the field is given by the formula $F_x(x, y, z) = -\frac{\partial U}{\partial x}$. Find F_x

$U(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$-\frac{\partial U}{\partial x} = (-1)\cdot\left(-\frac{1}{2}\right)GM(x^2+y^2+z^2)^{-3/2}\cdot 2x = \frac{GMx}{r^3}$

(b) $\star$ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2 + F_z^2}$. How does it decay as a function of r?

$|\vec{F}|^2 = F_x^2 + F_y^2 + F_z^2 = \left(\frac{GM}{r^3}\right)^2\cdot(x^2+y^2+z^2) = \frac{(GM)^2}{r^6}\cdot r^2 = \left(\frac{GM}{r^2}\right)^2$

so $|\vec{F}| = \frac{GM}{r^2}$

(8) The *entropy* of an ideal gas of N molecules at temperature T and volume V is

$$S(N,V,T) = Nk\log\left[\frac{VT^{1/(\gamma-1)}}{N\Phi}\right].$$

where k is *Boltzmann's constant* and γ, Φ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.

$$S = Nk\log\left(\frac{V}{N\Phi}\right) + Nk\log\left(T^{\frac{1}{\gamma-1}}\right) = Nk\log\left(\frac{V}{N\Phi}\right) + \frac{Nk}{\gamma-1}\log T$$

$$\frac{\partial S}{\partial T} = \frac{Nk}{\gamma-1}\frac{1}{T}, \text{ so } \quad C_V = \frac{Nk}{\gamma-1}$$

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write S as a function of N, P, T instead. Differentiating with respect to T **while keeping P constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$

$$S(N,P,T) = Nk\log\left[\frac{k\,T^{\gamma/\gamma-1}}{P\Phi}\right] \leftarrow V = \frac{NkT}{P}$$

$$= Nk\log\left[\frac{k}{P\Phi}\right] + Nk\cdot\frac{\gamma}{\gamma-1}\log T$$

so $C_P = T\frac{\partial S}{\partial T} = \frac{Nk\gamma}{\gamma-1}$

constant P

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f$. Evaluate:

(a) $\star\ h_{xx} = \frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(Ay^2 e^{Axy}\right) = A^2 y^3 e^{Axy}$

(b) $\star\ h_{xy} = \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left(Ay^2 e^{Axy}\right) = \left(2Ay + A^2 x y^2\right) e^{Axy}$

(c) $\star\ h_{yx} = \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left((1+Axy) e^{Axy}\right) = \left(Ay + Ay + A^2 x y^2\right) e^{Axy}$

(d) $\star\ h_{yy} = \frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left((1+Axy) e^{Axy}\right) = \left(Ax + Ax + A^2 x^2 y\right) e^{A}$

Facts $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for all ordinary functions

(10) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street
(say oriented toward the south), and let the y axis run across the street. Let $z = z(x, y)$ denote the height of
the street surface above sea level.

(a) ⋆ What does $\frac{\partial z}{\partial y} = 0$ say about the street?

The street is level

(b) ⋆ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

street is sloping up toward south ; grade is 15%.

(c) ⋆ You want to follow the street downhill. Which way should you go?

North, toward bus loop

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum.
What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?

Then we see max along both streets , so $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$