10. Differential Equations (8/11/2023)

Goals.

(1) Differential equations
  (a) What is a differential equation?
  (b) What isa solution to a differential equation?
  (c) Plugging ansatze into equations

Last Time. **Optimization**

(1) If f is defined on $[a,b]$, at x_0

(a) f has a local min if for x near x_0, $f(x) \geq f(x_0)$

x_0 can be endpoint (max) $(f(x) \leq f(x_0))$

(b) f has a global min, on $[a,b]$ if for all $x \in [a,b]$, $f(x) \geq f(x_0)$

(max) $(f(x) \leq f(x_0))$

Theorem: Let f be cts $[a,b]$ ("closed interval method")

(1) f has a global max/min on $[a,b]$

(2) Any local extremum occurs at one of critical pt
singular pt
endpoint

Equality in mathematics

Usually, when we write $A=B$ we are making an <u>assertion</u>: we claim two mathematical object are equal

Eg. $5+7=6+6$ (true) (assertions can be true or false)

$$\frac{d(x^2)}{dx}=x \quad \text{(false)}$$

Equations

Sometimes we interpret $A=B$ as an equation:
in that case A,B depend on unknown(s), ~~the~~ and A,B might be equal or not depending on the unknows
The solutions are the values for the unknown(s) that create true equalities

WS1

1. To check if $y=a$ solves an equation, replace/plugin a for y, see if we get an equality.
2. equations can have multiple solutions

Differential equations

1. Unkaowns will be **functions**
2. desired equality is of "
3. Equation involves derivatives of the unknown functions

otherwise same idea

Math 100A – WORKSHEET 10
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

$3^2 = 9 \neq 4$ so 3 is not a solution

$2^2 = 4$ so 2 is a solution

$y^2 = 4 \iff (y-2)(y+2) = 0$
all solutions: ± 2.

$3^2 = 3 \cdot 3$ so 3 is a solution

$2^2 = 4 \neq 6 = 3 \cdot 2$ so 2 isn't

$y^2 = 3y \iff y(y-3) = 0$
so solutions are 0, 3

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

$\frac{d(x^2)}{dx} = 2x \neq x^2$ X

not same function even if graphs intersect at som x

$\frac{d(e^x)}{dx} = e^x$ ✓

$\left(\frac{d(x^2)}{dx}\right)^2 = 4x^2 = 4 \cdot x^2$ ✓

$\left(\frac{d(e^x)}{dx}\right)^2 = e^{2x} \neq 4 \cdot e^x$ X

Date: 8/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(4) Which of the following (if any) is a solution of $\frac{dy}{dx} = \frac{x}{y}$

A. $y = -x$; B. $y = x+5$ C. $y = \sqrt{x^2+5}$

(3) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$;   B. $z(t) = t^2 + 2t + 1$

$$\frac{d(t^2)}{dt} + t^2 - 1 = 2t + t^2 - 1 \neq t^2$$ ~~but~~ (A not a solution)

but

$$\frac{d(t^2+2t+1)}{dt} + t^2 - 1 = 2t + 2 + t^2 - 1 = t^2 + 2t + 1 \quad \checkmark$$

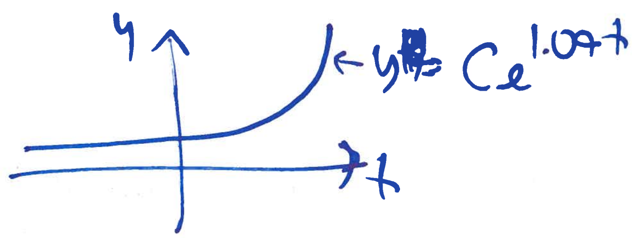
(B is a solution)

(write equation as $\frac{dz}{dt} - z = 1 - t^2$)

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

$y(t) = e^{1.04t}$ is a solution, in fact $y(t) = Ce^{1.04t}$ is the general solution (C arbitrary constant)

family o solutions covering al solutions



the particular solution with $y(0)=100$ has $C \cdot e^{1.04 \cdot 0} = C$

so $C = 100$, and the solution is $100e^{1.04t}$

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

Examples of Differential equations

(1) $\boxed{y' = ry}$ know: solution is $\boxed{y(t) = y(0)e^{rt}}$.
(general solution $\boxed{Ce^{rt}}$)

(2) F = ma : $\dfrac{d^2x}{dt^2} = \dfrac{1}{m} F(x, \dfrac{dx}{dt}, t)$ (Newton)

(3) Wave equation in musical instrument

Ansatz: Family of ~~solutions~~ function we guess might contain a true solutions

Family will depend on parameters, whether a member works will be an algebraic equation for the parameter values

(9) Consider the equation $\frac{dy}{dt} = a(y - b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by u?

$\frac{du}{dt} = \frac{dy}{dt} = a(y-b) = au$ | or $y = u + b$: $\frac{dy}{dt} = au$

so need $u(t) = Ce^{at}$

so $\boxed{y(t) = b + Ce^{at}}$

(b) What is the general solution for $u(t)$?

(c) What is the general solution for $y(t)$?

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?

$y(t)$ converges to b as $t \to \infty$, exponentially fast

(in fact $y(t) \equiv b$ is a solution: "steady state"
"equilibrium"
"fixed point")

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?

If $y(0) = y_0$ then $b + C \cdot e^0 = y_0$

so $C = y_0 - b$

$y(t) = b + (y_0 - b) \cdot e^{at}$

(10) Example: *Newton's law of cooling.* Suppose we place an object of temperature $T(0)$ in an environment of temperature T_{env}. It turns out that a good model for the temperature $T(t)$ of the object at time t is

$$\frac{dT}{dt} = -k\,(T - T_{\text{env}})$$

where $k > 0$ is a positive constant.

(a) Suppose $T(t) > T_{\text{env}}$. Is $T'(t)$ positive or negative? What if $T(t) < T_{\text{env}}$? Explain this in words.

(b) A body is found at 1:30am and its temperature is measured to be 32.5°C. At 2:30am its temperature is found to be 30.3°C. The temperature of the room in which the body was found is measured to be 20°C and we have no reason to believe the ambient temperature has changed. What was the time of death?

Use $u(t) = T(t) - 20°$, t = time in hours since 1:30am

By NLC, $u(t)$ decays exponentially:

$$u'(t) = -k\,u(t) \quad \text{for some } k,$$

$$u(t) = Ce^{-kt}$$

Know: $u(0) = 32.5 - 20 \Rightarrow C = 12.5 \quad (C = u(0))$

$u(1) = 30.3 - 20 \Rightarrow 12.5\,e^{-k} = 10.3$

$$\text{so } k = \log\frac{12.5}{10.3} \quad \left(\text{Sanity check: } \log\frac{12.5}{10.3} > 0\right)$$

At time of death expect:

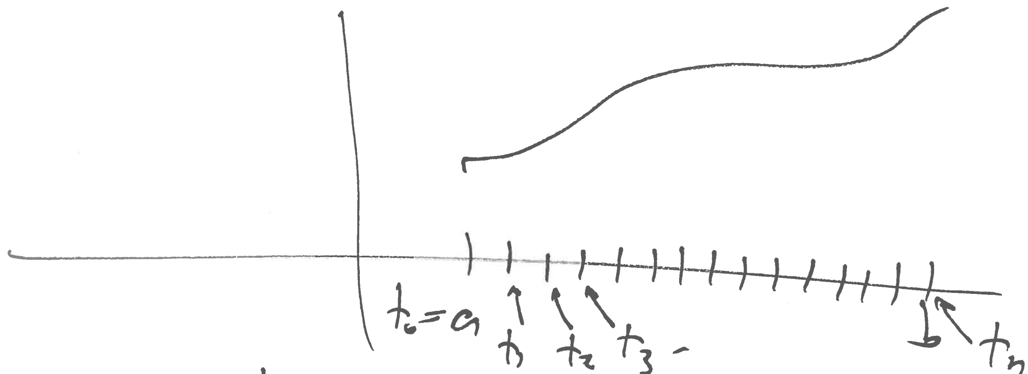
$$u(t) = 37° - 20° \quad \text{so} \quad 12.5 \cdot e^{-\left(\log\frac{12.5}{10.3}\right)t} = 17$$

$$\text{so} \quad t = -\frac{\log(17/12.5)}{\log(12.5/10.3)} \text{ hours}$$

(Sanity check: this is negative

Numerical solution

Idea: Want to solve $y' = F(y,t)$ on $[a,b]$ by computer



① divide interval into n sub-intervals of length $h = \Delta x = \frac{b-a}{n}$.

② compute guesses $y_0, y_1, y_2, \ldots y_k$ for $y(t_k)$

y_0 is given ("initial" condition)

given $y_k \approx f(t_k)$, $\Rightarrow$ guess $F(y_k, t)$ for $y'(t_k)$

now guess y_{k+1} for $y(t_{k+1}) = y(t_k + h)$

$$\boxed{y_{k+1} \approx y_k + F(y_k, t_k) \cdot h}$$

(Euler Scheme)