Math 100A – SOLUTIONS TO WORKSHEET 6
APPLICATIONS OF THE CHAIN RULE

1. Review

(1) Differentiate
  (a) ⋆ $e^{\sqrt{\cos x}}$
  Solution: We repeatedly apply the chain rule:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x}\\ &= e^{\sqrt{\cos x}}\frac{1}{2\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\cos x\\ &= -e^{\sqrt{\cos x}}\frac{\sin x}{2\sqrt{\cos x}}\,.\end{aligned}$$

(2) (Final, 2014) ⋆ Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.
  Solution: By the logarithmic differentiation rule we have
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = y\frac{\mathrm{d}\log y}{\mathrm{d}x} &= x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x \cdot \log x\right)\\ = x^{\log x}\left(2\log x\cdot\frac{1}{x}\right) &= 2\log x \cdot x^{\log x - 1}\,.\end{aligned}$$

2. Implicit Differentiation

(3) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2,6)$.
  Solution: Differentiating with respect to x we find $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 2$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2+1}{y}$. In particular at the point $(2,6)$ the slope is $\frac{25}{6}$ and the line is
$$y = \frac{25}{6}(x-2) + 6\,.$$

(4) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1,1)$.
  Solution: Differentiating with respect to x we find $y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ along the curve. Setting $x = y = 1$ we find that, at the indicated point,
$$3 + 3\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,1)} = 0$$
so
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,1)} = -1\,.$$

(5) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0,1)$.
  Solution: Differentiating with respect to x we find $y' + \cos y - x\sin y \cdot y' = -\sin x$, so that $y' = -\frac{\sin x + \cos y}{1 - x\sin y} = \frac{\sin x+\cos y}{x\sin y - 1}$. Setting $x = 0$, $y = 1$ we get that at that point $y' = \frac{\cos 1}{-1} = -\cos 1$.

(6) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).
  Solution: Differentiating with respect to x we find $5x^4 + 5y^4y' = 0$, so that $y' = -\frac{x^4}{y^4}$. Differentiating again we find
$$y'' = -\frac{4x^3}{y^4} + \frac{4x^4y'}{y^5} = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9}\,.$$

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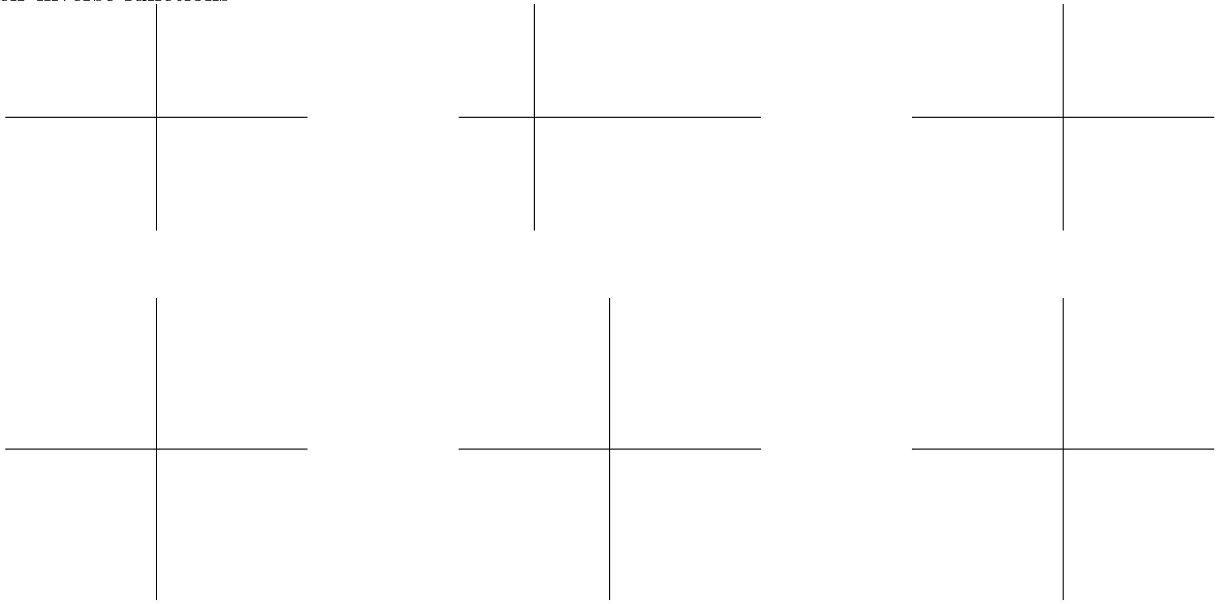
3. Inverse trig functions

(7) Draw on the following axes graphs of $\sin\theta$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta$ on $[0,\pi]$ and $\tan\theta$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then of their inverse functions



(8) Evaluation

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$ and $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.
Solution: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. Also $\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9\pi}{11}\right) = \sin\left(\pi - \frac{9\pi}{11}\right) = \sin\left(\frac{2\pi}{11}\right)$ and $\frac{2\pi}{11} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right) = \frac{2\pi}{11}$.

(b) (Final 2015) Simplify $\sin(\arctan 4)$
Solution: Consider the right-angled triangle with sides 4, 1 and hypotenuse $\sqrt{1+4^2} = \sqrt{17}$. Let θ be the angle opposite the side of length 4. Then $\tan\theta = 4$ and $\sin\theta = \frac{4}{\sqrt{17}}$ so $\sin(\arctan 4) = \sin\theta = \frac{4}{\sqrt{17}}$.

(c) Find $\tan(\arccos(0.4))$
Solution: Consider the right-angled triangle with sides 0.4, $\sqrt{1-0.4^2}$ and hypotenuse 1. Let θ be the angle between the side of length 0.4 and the hypotenuse. Then $\cos\theta = \frac{0.4}{1} = 0.4$ and $\tan\theta = \frac{\sqrt{1-0.4^2}}{0.4} = \frac{\sqrt{0.84}}{0.4} = \sqrt{\frac{0.84}{0.16}} = \sqrt{5.25}$.

(9) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x)$
Solution: Let $\theta = \arctan x$. Then $x = \tan\theta$ so by the chain rule $1 = \frac{dx}{dx} = \frac{d\tan\theta}{dx} = \frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{dx} = (1+\tan^2\theta)\frac{d\theta}{dx}$ so

$$\frac{d(\arctan x)}{dx} = \frac{d\theta}{dx} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+x^2}\,.$$

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$
Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x)) = \frac{2}{\sqrt{1-4x^2}}\,.$$

Alternatively, let $\theta = \arcsin 2x$, so that $\sin\theta = 2x$. Differentiating both sides we get

$$\cos\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x} = 2$$

so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{2}{\cos\theta} = \frac{2}{\sqrt{1-\sin^2\theta}} = \frac{2}{\sqrt{1-4x^2}}\,.$$

(c) Find the line tangent to $y = \sqrt{1+(\arctan(x))^2}$ at the point where $x = 1$.
Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$, the chain rule gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+(\arctan(x))^2} &= \frac{1}{2\sqrt{1+(\arctan(x))^2}}\cdot 2\arctan(x)\cdot\frac{1}{1+x^2}\\ &= \frac{\arctan x}{(1+x^2)\sqrt{1+(\arctan(x))^2}}\,.\end{aligned}$$

Now $\arctan 1 = \frac{\pi}{4}$ so the line is

$$y = \frac{\pi}{8\sqrt{1+\frac{\pi^2}{16}}}(x-1) + \sqrt{1+\frac{\pi^2}{16}}\,.$$

(d) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?
Solution: From the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(e^{5x}\right) = \frac{1}{\sqrt{1-e^{10x}}}5e^{5x} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}\,.$$

The function y itself is defined when $-1 \le e^{5x} \le 1$, that is when $5x \le 0$, that is when $x \le 0$. The derivative is defined when $-1 < e^{10x} < 1$, that is when $x < 0$. The point is that since $\sin\theta$ has horizontal tangents at $\pm\frac{\pi}{2}$, $\arcsin x$ has vertical tangents at ± 1.
Solution: We can write the identity as $\sin y = e^{5x}$ and differentiate both sides to get $y'\cos y = 5e^{5x}$ so that

$$y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1-\sin^2 y}} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}\,.$$

4. Related Rates

(10) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $\left(1,\sqrt{3}\right)$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.
Solution: We differentiate along the curve with respect to time, finding

$$2y\frac{\mathrm{d}y}{\mathrm{d}t} = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

Plugging in $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$, $x = 1$, $y = \sqrt{3}$ we find: $2\sqrt{3} = 5\frac{\mathrm{d}x}{\mathrm{d}t}$ so at that time we have

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2\sqrt{3}}{5}\,.}$$

(11) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.
(a) The drain is clogged, and is filling up with rainwater at the rate of $5\mathrm{m}^3/\mathrm{min}$. How fast is the water rising when its height is 5m?
Solution: The water fills a conical volume inside the drain. Suppose that at time t the height of the water is $h(t)$ and the radius at the surface of the water is $r(t)$. Then by similar triangles

$$\frac{r(t)}{h(t)} = \frac{1}{6}\,.$$

We therefore have $r(t) = \frac{h(t)}{6}$. The volume of the water is therefore

$$V(t) = \frac{1}{3}\pi r^2 h = \frac{\pi}{108}h^3(t)\,.$$

Differentiating we find

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{36} h^2(t) \frac{\mathrm{d}h}{\mathrm{d}t}\,.$$

In particular, if $\frac{\mathrm{d}V}{\mathrm{d}t} = 5\mathrm{m}^3/\mathrm{min}$ and $h = 5\mathrm{m}$ then

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{36 \cdot 5}{\pi \cdot 5^2} = \frac{36}{5\pi} \frac{\mathrm{m}}{\mathrm{min}}\,.$$

(b) The drain is unclogged and water begins to drain at the rate of $(5 + \frac{\pi}{4})\mathrm{m}^3/\mathrm{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Solution: We are now given $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\pi}{4} \frac{\mathrm{m}^3}{\mathrm{min}}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = -1 \frac{\mathrm{m}}{\mathrm{min}}$. Thus at the given time

$$h(t) = \sqrt{\frac{36 \frac{\mathrm{d}V}{\mathrm{d}t}}{\pi \frac{\mathrm{d}h}{\mathrm{d}t}}} = \sqrt{\frac{-36\pi}{4\pi(-1)}} = \sqrt{9} = 3\mathrm{m}\,.$$