Math 100A – SOLUTIONS TO WORKSHEET 13
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

(1) $\star$How many critical points does $f(x,y) = x^2 - x^4 + y^2$ have?
Solution: $\frac{\partial f}{\partial x}(x,y) = 2x - 4x^3 = 2x(1-2x^2)$ while $\frac{\partial f}{\partial y} = 2y$. Thus $\frac{\partial f}{\partial y} = 0$ only when $y = 0$ while $\frac{\partial f}{\partial x} = 0$ when $x \in \left\{0, \pm\frac{1}{\sqrt{2}}\right\}$. Thus there are three critical points: $(0,0)\,, \left(\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0\right)$.

(2) $\star$Find the critical points of $f(x,y) = x^2 - x^4 + xy + y^2$.
Solution: Now $\frac{\partial f}{\partial x}(x,y) = 2x - 4x^3 + y$ while $\frac{\partial f}{\partial y} = x + 2y$. At a critical point we have $\frac{\partial f}{\partial y} = 0$ so $y = -\frac{1}{2}x$ and also $\frac{\partial f}{\partial x}(x,y) = 0$ so $2x - 4x^3 + y = 0$. Substituting $y = -\frac{1}{2}x$ we get $\frac{3}{2}x - 4x^3 = 0$ or $-4x\left(x^2 - \frac{3}{8}\right) = 0$ so we have a critical point when $x \in \left\{0, \pm\sqrt{\frac{3}{8}} = \pm\frac{1}{2}\sqrt{\frac{3}{2}}\right\}$ and hence at the points $\left\{(0,0)\,, \left(\frac{1}{2}\sqrt{\frac{3}{2}}, -\frac{1}{4}\sqrt{\frac{3}{2}}\right), \left(-\frac{1}{2}\sqrt{\frac{3}{2}}, \frac{1}{4}\sqrt{\frac{3}{2}}\right)\right\}$.

(3) (MATH 105 Final, 2013) $\star$ Find the critical points of $f(x,y) = xye^{-2x-y}$.
Solution: $\frac{\partial f}{\partial x}(x,y) = ye^{-2x-y} - 2xye^{-2x-y} = y(1-2x)e^{-2x-y}$ while $\frac{\partial f}{\partial y}(x,y) = xe^{-2x-y} - xye^{-2x-y} = x(1-y)e^{-2x-y}$. Since $e^{-2x-y} \neq 0$ everywhere, the critical points are the solutions to the system of equations

$$\begin{cases} y(1-2x) = 0 \\ x(1-y) = 0 \end{cases}.$$

Starting with the second equation we either have $x = 0$ or $y = 1$. In the first case the first equation reads $y = 0$ and we get the critical point $(0,0)$. In the second case the first equation reads $1 - 2x = 0$ and we get the critical point $\left(\frac{1}{2}, 1\right)$.

(4)
(a) $\star\star$ Let $f(x,y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").
Solution: $\frac{\partial f}{\partial x} = 8x$ and $\frac{\partial f}{\partial y} = 16y$. The only point where both vanish is where $x = y = 0$ where $f(0,0) = 7$. Since $4x^2 + 8y^2 \geq 0$ for all x, y we have $f(x,y) \geq 7$ for all x, y so this point is the global minimum, and in particular a local minimum.
(b) (MATH 105 Final, 2017) $\star\star$ Let $f(x,y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").
Solution: $\frac{\partial f}{\partial x} = -8x$ and $\frac{\partial f}{\partial y} = 16y$. The only point where both vanish is where $x = y = 0$ where $f(0,0) = -3$. We have a local *maximum* along the x axis (for constant y the parabola $-4x^2 + \left(8y^2 - 3\right)$ is concave down) but a local *minimum* along the y axis (for constant x the parabola $8y^2 - \left(4x^2 + 3\right)$ is concave up), so this is a saddle point.

(5) $\star$ Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.
Solution: Since $\frac{\partial f}{\partial x} = e^{-x-y}(7 - 7x - 3y - 2y^2)$ and $\frac{\partial f}{\partial y} = e^{-x-y}(3 + 4y - 7x - 3y - 2y^2)$ the critical points are at

$$\begin{cases} 7x + 3y + 2y^2 & = 7 \\ 7x + 3y + 2y^2 & = 3 + 4y\,. \end{cases}$$

Date: 29/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

At a solution of this system we must have $3+4y=7$ so $y=1$ and then $7x=7-3y-2y^2$ forces $x=\frac{2}{7}$, so the only critical point is at $\left(\frac{2}{7},1\right)$.

2. Optimization

(6) $\star\star$Find the minimum of $f(x,y)=2x^2+3y^2-4x-5$:

(a) on the rectangle $0\leq x\leq 2$, $-1\leq y\leq 1$.

Solution: We have $\frac{\partial f}{\partial x}=4x-4=4(x-1)$ and $\frac{\partial f}{\partial y}=6y$ so the only critical point is at $(1,0)$ where $f(1,0)=-7$. We now examine the boundary. If $y=\pm 1$ we have

$$f(x,\pm 1)=2x^2-4x-2$$

which has $f(0,\pm 1)=-2$, $f(2,\pm 1)=-2$ and a critical point at $x=1$ where $f(1,\pm 1)=-4$ so the minimum on those edges is -4. If $x=0$ we have

$$f(0,y)=3y^2-5$$

which clearly has a minimum $f(0,0)=-5$. If $x=2$ we have $f(2,y)=3y^2-5$ and the same conclusion follows.

Bottom line: the minimum is -7 and occurs at the critical point

Solution: We have $f(x,y)=2(x^2-2x+1)+3y^2-7=2(x-1)^2+3y^2-7$ so the minimum is at $(1,0)$.

(b) on the rectangle $2\leq x\leq 3$, $-1\leq y\leq 1$.

Solution: There are now no critical points in the rectangle, so the maximum occurs on the boundary. As before when $y=\pm 1$ we have

$$f(x,\pm 1)=2x^2-4x-2$$

which has $f(2,\pm 1)=-2$ and $f(3,\pm 1)=4$ and no critical points ($\frac{\partial f}{\partial x}$ only vanishes at $x=1$) so the minimum on these edges is -2. Similarly

$$f(2,y)=3y^2-5$$

has its minimum at $y=0$ where $f(2,0)=-5$ while $f(3,y)=1+3y^2$ which for $-1\leq y\leq 1$ has its minimum value 1 at $y=0$. It follows that the minimum value is 5 attained at $(2,0)$.

(7) Find the maximum of $(7x+3y+2y^2)e^{-x-y}$ for $x\geq 0$, $y\geq 0$,

Solution: We start with the boundary. If $y=0$ we have $f(x,0)=7xe^{-x}$, the derivative of which is $7e^{-x}-7xe^{-x}=7(1-x)e^{-x}$ which only vanishes at $x=1$. The maximum is then at $x=1$ where the value is $\frac{7}{e}$. If $x=0$ we get $f(0,y)=(3y+2y^2)e^{-y}$ with derivative $(3+4y-3y-2y^2)e^{-y}=-\left(2y^2-y-3\right)e^{-y}$. This vanishes at $y=\frac{1\pm\sqrt{25}}{4}=\frac{3}{2},-1$, so at $y=\frac{3}{2}$. Since $f(0,0)=0$, $f(0,\frac{3}{2})=9e^{-3/2}>0$ and $f(0,y)$ is negative for large y, the maximum on this boundary is at $9e^{-3/2}$. Finally the function tends to 0 if $x\to\infty$ or $y\to\infty$ (the exponential always wins) so there will be a maximum which, if it occurs at the interior, must occur at a critical point. We already saw that the only critical point is at $\left(\frac{2}{7},1\right)$, and evaluation gives $f(\frac{2}{7},1)=7e^{-9/7}<\frac{7}{e}$. The maximum is therefore at the larger of the boundary values. Now

$$\left(\frac{7}{e}\Big/\frac{9}{e^{3/2}}\right)^2=\frac{7^2e}{9^2}>\frac{49\cdot 2}{81}>1$$

so $\frac{7}{e}$ is the largest value, hence the maximum. (With a calculator we could also check that $\frac{7}{e}\approx 2.58$, $\frac{9}{e^{3/2}}\approx 2.01$, and $\frac{7}{e^{9/7}}\approx 1.94$).

(8) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C=3t^2+\sqrt{t}\cdot q$. At price p the company can sell $q=\frac{t-p}{3}$ widgets.

(a) Write an expression for the profit function $f(q,t)$.

Solution: To sell q widgets the price must be $p=t-3q$, so the revenue will be $R=qp=tq-3q^2$ and the profit will be

$$f(q,t)=R-C=tq-3q^2-3t^2+\sqrt{t}\cdot q\,.$$

(b) How many widgets of what quality should the company make to maximize profits?
Solution: We need to maximize

$$f(q,t) = tq - 3q^2 - 3t^2 + \sqrt{t}\cdot q\,.$$

Now $\frac{\partial f}{\partial q} = t - 6q + \sqrt{t}$ while $\frac{\partial f}{\partial t} = q\left(1 + \frac{1}{2\sqrt{t}}\right) - 6t$. From the first equation we find that at fixed quality we maximize profits at $q = \frac{t+\sqrt{t}}{6}$. As $t \to \infty$ $q \sim \frac{t}{6}$ so

$$\begin{aligned} f(q,t) &\sim t\cdot\frac{t}{6} - 3\left(\frac{t}{6}\right)^2 - 3t^2 + \sqrt{t}\frac{t}{6} \\ &\sim -\left(3 - \frac{1}{6}\right)t^2 \to -\infty \end{aligned}$$

so there is a limit to the qualities at which we will make a profit. Conversely at quality 0 we have $f(q,0) = -3q^2 \leq 0$ so we must have some positive quality to make a profit, and the maximum will occur at a critical point. Plugging $q = \frac{t+\sqrt{t}}{6}$ into $\frac{\partial f}{\partial t} = 0$ we get the equation

$$\frac{1}{6}\left(t + \sqrt{t}\right)\left(1 + \frac{1}{2\sqrt{t}}\right) - 6t = 0$$

that is

$$t + \frac{3}{2}\sqrt{t} + \frac{1}{2} = 36t$$

or

$$70\left(\sqrt{t}\right)^2 - 3\sqrt{t} - 1 = 0$$

which has the solution

$$\begin{aligned} \sqrt{t} &= \frac{3 \pm \sqrt{9 + 4\cdot 70}}{2\cdot 70} = \frac{3 \pm \sqrt{289}}{140} \\ &= \frac{20}{140} = \frac{1}{7} \end{aligned}$$

since we must have $\sqrt{t} > 0$. At this value we have $q = \frac{8}{49\cdot 6} = \frac{4}{3\cdot 49}$ and $f(q,t) = \frac{7}{3\cdot 49^2} = \frac{1}{3\cdot 343} > 0$, so this is indeed the maximum.

(9) Find the maximum and minimum values of $f(x,y) = -x^2 + 8y$ in the disc $R = \left\{x^2 + y^2 \leq 25\right\}$.

Solution: $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = 8$, so f has no critical points in the interior of the disc (or anywhere, for that matter), and the minimum and maximum must occur on the boundary, where $x^2 + y^2 = 25$, so $-x^2 = y^2 - 25$ and (**only there**)

$$f(x,y) = y^2 + 8y - 25 = (y+4)^2 - 41\,.$$

The minimum is therefore at the point(s) where y is closest to -4 and the maximum is where they are furthest away. Since $(\pm 3, -4)$ are on the circle $x^2 + y^2 = 5$ the minimum is -41 attained at $(\pm 3, -4)$. On the circle we have $-5 \leq y \leq 5$ so the maximum of $(y+4)^2$ is where $y = 5$ (and $x = 0$). Thus the maximum is 40 attained at $(0,5)$.

(10) (MATH 105 final, 2015) Find the maximum and minimum values of $f(x,y) = (x-1)^2 + (y+1)^2$ in the disc $R = \left\{x^2 + y^2 \leq 4\right\}$.

Solution: We have $\frac{\partial f}{\partial x} = 2(x-1)$ and $\frac{\partial f}{\partial y} = 2(y+1)$ so the only critical point is $(1,-1)$ where $f(1,-1) = 0$. Since $f(x,y) \geq 0$ for all x, y this must be the global minimum. The maximum must therefore occur on the boundary where $x^2 + y^2 = 4$. There

$$f(x,y) = x^2 + y^2 - 2x + 1 + 2y + 1 = 6 - 2x + 2y\,.$$

Now *along the curve* $x^2 + y^2 = 4$ we have $2y\frac{dy}{dx} + 2x = 0$ so $\frac{dy}{dx} = -\frac{x}{y}$. *Along that curve* we thus have

$$\frac{df}{dx} = -2 + 2\frac{dy}{dx} = -2 - \frac{2x}{y}\,.$$

From the point of view of optimization on the boundary we then have a critical point where $\frac{x}{y} = -1$ that is $x = -y$ and a singular point where $y = 0$. Now $x = -y$ means $x^2 = y^2 = 2$ so the points are $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. When $y = 0$ we have $x = \pm 2$. We now evaluate f at these points:

$$f(-2,0) = 10 \qquad f(2,0) = 2$$

$$f\left(\sqrt{2}, -\sqrt{2}\right) = 6 - 4\sqrt{2} \qquad f\left(-\sqrt{2}, \sqrt{2}\right) = 6 + 4\sqrt{2}$$

and since $\sqrt{2} > 1$ we see that the maximum is $6 + 4\sqrt{2}$ at $(-\sqrt{2}, \sqrt{2})$.

(11) (The inequality of the means) We calculate the maximum of $f(x,y,z) = xyz$ on the domain $x+y+z = 1$, $x, y, z \geq 0$.

(a) Explain why it's enough to find the maximum of $g(x,y) = xy(1-x-y)$ on the domain $x \geq 0, y \geq 0$, $x + y \leq 1$.

(b) Find the critical points of g in the interior of the domain, and the values of g at those points.
Solution: We have $\frac{\partial g}{\partial x} = y(1-x-y) - xy = y(1-2x-y)$ so $\frac{\partial f}{\partial y} = x(1-2y-x)$. Since $x, y \neq 0$ inside the domain the critical points are the solutions of

$$\begin{cases} 2x + y &= 1 \\ x + 2y &= 1 \end{cases},$$

and it's easy to check that the only solution is $x = y = \frac{1}{3}$ where $g\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$.

(c) What is the boundary of the domain of g? What is the maximum there?
Solution: The edges of the triangle are $x = 0, 0 \leq y \leq 1$, $y = 0, 0 \leq y \leq 1$, and the line $x + y = 1$ and on all of them we have $g \equiv 0$ so the maximum is 0.

(d) What is the maximum of g?
Solution: The largest value is $\frac{1}{27}$, attained at $\left(\frac{1}{3}, \frac{1}{3}\right)$.

(e) Show that for all $X, Y, Z \geq 0$ we have $(XYZ)^{1/3} \leq \frac{X+Y+Z}{3}$ (the "inequality of the means"). Hint: define $x = \frac{X}{X+Y+Z}$, $y = \frac{Y}{X+Y+Z}$, $z = \frac{Z}{X+Y+Z}$ and apply the previous result.

3. Lagrange multipliers (MATH 100C)

(11) (MATH 105 final, 2017) Use the mConstrained optimizationethod of Lagrange Multipliers to find the maximum value of the utility function $U = f(x,y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x,y) = 50x + 100y - 500,000 = 0$, where $x \geq 0$ and $y \geq 0$.

Solution: If $x = 0$ or $y = 0$ we have $f(x,y) = 0$ while if $x, y > 0$ we have $f(x,y) > 0$ so the maximum must be in the interior of the domain (and occur at a critical point). By the method of Lagrange Multipliers the maximum occurs at a point x, y where

$$\begin{cases} 4x^{-3/4}y^{3/4} = 50\lambda \\ 12x^{1/4}y^{-1/4} = 100\lambda \\ 50x + 100y - 500,000 = 0\,. \end{cases}$$

If $\lambda = 0$ then either $x = 0$ or $y = 0$ by the first two equations, Constrained optimizationwhich isn't the case, so $\lambda \neq 0$ and we can divide the second equation by the first. We get:

$$3\frac{x}{y} = 2\,,$$

that is $3x = 2y$. Writing the equation of the constraint as $x + 2y = 10,000$ we see that we must have $4x = 10,000$ so $x = 2,500$ and $y = \frac{3x}{2} = 3,750$. Since this is the only solution it must be the maximum, and the value is

$$\begin{aligned} f(2500, 3750) &= 16 \cdot \left(\frac{10^4}{4}\right)^{1/4} \left(3\frac{10^4}{8}\right)^{3/4} \\ &= 2^4 \frac{10}{\sqrt{2}} \cdot 10^3 \cdot 3^{3/4} \cdot 2^{-9/4} = 10^4 \cdot 2^{\frac{5}{4}} \cdot 3^{3/4} \\ &= 20,000 \times 2^{1/4}3^{3/4}\,. \end{aligned}$$

(12) Labour-Leisure model: a person can choose to spend L hours a day not working ("leisure"), working $24 - L$ hours with way w. Suppose their fixed income is V dollars per day. Their consumption of goods is them $C = w(24 - L) + V$, equivalenly $C + wL = 24w + V$ (here C, L are variables while w, V are constants). If their utility function is $U = U(C, L)$ find a system of equations for maximum utility.

Solution: We need to maximize $U(C, L)$ subject to the *budget constraint* $C + wL = 24w + V$, so we get the system

$$\begin{cases} \frac{\partial U}{\partial C} & = \lambda \\ \frac{\partial U}{\partial L} & = \lambda w \\ C + wL & = 24w + V \, . \end{cases}$$