Math 100:V02
Problem Set 1: The Square Well

Due: Friday April 12th, 2024

Instructions

- Please submit *typeset* solutions through Canvas.
- Solutions must be written in complete English sentences: it's not enough to write a sequence of formulas.
- Do not hesitate to ask for help, whether in-person or on Piazza.

Problem The motion of an electron with energy E is moving along a 1-dimensional wire is described by a *wavefunction* $f(x)$. There is an impurity in the wire, between $-L \leq x \leq L$, so that the function f satisfies the following differential equation:

$$f''(x) = \begin{cases} -Ef(x) & |x| < L \\ (V-E)f(x) & |x| > L \end{cases}.$$

1. Assume $0 < E < V$. Find $\alpha, \beta > 0$ so that

$$f(x) = \begin{cases} \cos(\alpha x) & |x| < L \\ Ae^{-\beta|x|} & |x| > L \end{cases}$$

   is a solution to the equation for $x \neq |L|$.

2. Both f and f' must be continuous at $x = L$ for a solution to the equations. Derive two independent equations that E and A must satisfy for this to be true.

3. Show that the continuity conditions require the energy to satisfy $\tan\left(\sqrt{E} \cdot L\right) = \sqrt{\frac{V-E}{E}}$. In particular, not every E can be the energy of the electron (this phenomenon is known as "quantization" of the energy).

4. Find $\frac{dE}{dV}$ in terms of E, V (and the fixed length L). Briefly describe what happens to the energy level E as the effect of the impurity increases?

Extra practice (not for submission) Repeat the previous problem but using the solution

$$f(x) = \begin{cases} \sin(\sqrt{E}x) & |x| < L \\ B\,\mathrm{sgn}(x)e^{-\sqrt{V-E}|x|} & |x| > L \end{cases}$$

instead. Here $\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$. The equation you derive in part 3 will be slightly different, and this will effect part 4.