Math 100, Lecture 7, 30/1/2024

Last time: exponentials: $C\cdot b^x$. Ce^{rx}.

$\boxed{\frac{d}{dx}(b^x) = (\log b)\cdot b^x}$ $\boxed{\frac{d\sin\theta}{d\theta} = \cos\theta, \quad \frac{d\cos\theta}{d\theta} = -\sin\theta}$

$\frac{d}{dx}(Ce^{rx}) = Cre^{rx} \Rightarrow y(x) = Ce^{rx}$ solves the differential equation $y' = ry$.

Ex. $\frac{d\tan\theta}{d\theta} = 1 + \tan^2\theta$. $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$

Know $\sin\theta, \cos\theta, \tan\theta$ for θ multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

Today: Chain rule

Handles differentiation of compositions of functions:

$$(f\circ g)(x) \overset{\text{def}}{=} f(g(x))$$

Math 100:V02 – WORKSHEET 7
THE CHAIN RULE

1. The Chain Rule

$\sin(y+h) \neq \sin y + \sin h$

(1) We know $\frac{d}{dy}\sin y = \cos y$.

(a) Expand $\sin(y+h)$ to linear order in h. Write down the linear approximation to $\sin y$ about $y = a$.

$\sin(y+h) \approx \sin y + (\cos y)\cdot h$

$\sin y \approx \sin a + \cos a \cdot (y-a)$

(b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

$F(x+h) = \sin(3(x+h)) = \sin(3x+3h) \approx \sin(3x) + \cos(3x)\cdot 3h$

$\approx F(x) + (\cos(3x)\cdot 3)h$

So $F'(x) = \cos(3x)\cdot 3$

step 1: make linear approx

step 2: read off coeff of h

Date: 30/1/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Say $F(x) = f(y)$ where $y = cx$

if $f(y+h) \approx f(y) + f'(y) \cdot h$

then

$$F(x+h) = f(c(x+h)) \approx f(cx+ch)$$
$$\approx f(cx) + f'(cx) \cdot ch$$
$$= F(x) + (f'(cx) \cdot c) \cdot h$$

so $\boxed{\frac{d}{dx}(f(cx)) = f'(cx) \cdot c}$

$$\frac{d\,f(cx)}{dx} = \underbrace{\frac{d(f(cx))}{d(cx)}}_{\frac{df(y)}{dy} \text{ if } y=cx} \cdot \underbrace{\frac{d(cx)}{dx}}_{c}$$

(2) Write each function as a composition and differentiate

(a) e^{3x} here $e^{3x} = e^{y}$ $y=3x$

$$\frac{d(e^y)}{dy} = e^y \qquad \frac{d(e^y)}{dx} = e^y \cdot 3 = e^{3x} \cdot 3$$

(b) $\sqrt{2x+1}$ $= \sqrt{y}$ where $y = 2x+1$

$$\frac{d(\sqrt{y})}{dy} = \frac{1}{2\sqrt{y}} \Rightarrow \frac{d(\sqrt{2x+1})}{dx} = \frac{1}{2\sqrt{2x+1}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$$

In general, if $F(x) = f(g(x)) = f(y)$, $y = g(x)$
what is $F'(x)$?

have $\quad g(x+h) \approx \underbrace{g(x)}_{y} + \underbrace{g'(x) \cdot h}_{\Delta y}$

so $\quad F(x+h) = f(g(x+h)) \approx f(g(x) + g'(x)h)$

if we work to 1st order
can replace g with its
linear approx

linear approx of f

$\approx f(g(x)) + f'(g(x)) \cdot g'(x)h$

$= F(x) + (f'(g(x)) \cdot g'(x))h$

so $\boxed{\dfrac{d(f(g(x)))}{dx} = \left.\dfrac{df}{dy}\right|_{y=g(x)} \cdot \dfrac{dg}{dx}}$ $\quad (f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\boxed{\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}}$$

(c) (Final, 2015) $\sin(x^2)$

① write $\sin(x^2) = \sin\theta$ with $\theta = x^2$

so $\frac{d(\sin(x^2))}{dx} = \frac{d(\sin\theta)}{dx} = \frac{d(\sin\theta)}{d\theta} \cdot \frac{d\theta}{dx} = \cos\theta \cdot 2x$
$= \cos(x^2) \cdot 2x$

② $\frac{d(\sin(x^2))}{dx} = \frac{d(\sin(x^2))}{d(x^2)} \cdot \frac{d(x^2)}{dx} = \cos(x^2) \cdot 2x$

③ $(\sin(x^2))' = \cos(x^2) \cdot 2x$

(d) $(7x + \cos x)^n$.

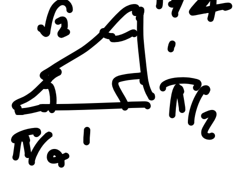
$$\frac{d}{dx}(7x+\cos x)^n \underset{\substack{\uparrow \\ \text{power law} \\ \text{rule}}}{=} n(7x+\cos x)^{n-1} \cdot \underbrace{\frac{d(7x+\cos x)}{dx}}_{\text{chain rule}}$$

$$= n(7x+\cos x)^{n-1} \cdot (7 - \sin x)$$

(3) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

by the chain rule. $f'(x) = g'(2\sin x) \cdot 2\cos x$

so $f'\left(\frac{\pi}{4}\right) = g'\left(2\sin\frac{\pi}{4}\right) \cdot 2\cos\left(\frac{\pi}{4}\right)$



$= g'(\sqrt{2}) \cdot \sqrt{2} = \sqrt{2} \cdot \sqrt{2} = 2$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$

(4) Differentiate

(a) $7x + \cos(x^n)$

$$(7x + \cos(x^n))' = 7 - \sin(x^n) \cdot nx^{n-1}$$

(b) $e^{\sqrt{\cos x}}$

$$\frac{d(e^{\sqrt{\cos x}})}{dx} = \frac{d(e^{\sqrt{\cos x}})}{d\sqrt{\cos x}} \cdot \frac{d(\sqrt{\cos x})}{d\cos x} \cdot \frac{d\cos x}{dx}$$

$$= e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)$$

(c) (Final 2012) $e^{(\sin x)^2}$

$$\left(e^{(\sin x)^2}\right)' = e^{(\sin x)^2} \cdot 2\sin x \cdot \cos x$$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

say $y = \log x$ want $\frac{dy}{dx}$.

$y = \log x$ if $e^y = x$ so $\frac{d(e^y)}{dx} = \frac{dx}{dx}$

so $1 = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \cdot \frac{dy}{dx} = e^y \cdot \frac{dy}{dx}$

so $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ $\Rightarrow$ $\boxed{\frac{d(\log x)}{dx} = \frac{1}{x}}$

2. Differentiating logarithms

(6) $\log\left(e^{10}\right) =$ $\log(2^{100}) =$

(7) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2+3t\right) =$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^2\log(1+x^2) =$ $\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$

(8) (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

(9) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) $\star$ x^n

(b) x^x

(c) $(\log x)^{\cos x}$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

3. More problems

(10) Let $f(x) = g(x)^{h(x)}$. Find a formula for f' in terms of g' and h'.

(11) Let $f(\theta) = \sin^2\theta + \cos^2\theta$. Find $\frac{df}{d\theta}$ without using trigonometric identities. Evaluate $f(0)$ and conclude that $\sin^2\theta + \cos^2\theta = 1$ for all θ.

(12) ("Inverse function rule") suppose $f(g(x)) = x$ for all x.

(a) Show that $f'(g(x)) = \frac{1}{g'(x)}$.

(b) Suppose $g(x) = e^x$, $f(y) = \log y$. Show that $f(g(x)) = x$ and conclude that $(\log y)' = \frac{1}{y}$.

(c) Suppose $g(\theta) = \sin\theta$, $f(x) = \arcsin x$ so that $f(g(\theta)) = \theta$. Show that $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

(13) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 1)$.