① Find line tangent to

$$3x^2y + y^3 = (x+y)^2$$

at $(1,1)$

Solution: As (x,y) varies on curve, both sides have same derivative (they are equal)

So along the curve

$$6xy + 3x^2\frac{dy}{dx} = 2(x+y)\left(1+\frac{dy}{dx}\right)$$

So at $(1,1)$: $\quad 6 + 3\frac{dy}{dx} = 2\cdot 2\left(1+\frac{dy}{dx}\right)$

so $\quad \left.\frac{dy}{dx}\right|_{(1,1)} = 2$

So the line is $\quad y = 2(x-1)+1$

② $y'(t) = \dfrac{Ae^t(Ae^t+1) - Ae^t(Ae^t-1)}{(Ae^t+1)^2} =$

$$= \frac{2Ae^t}{(Ae^t+1)^2}$$

For this Ansatz,

$$1-y^2 = 1 - \frac{(Ae^t-1)^2}{(Ae^t+1)^2} = \frac{(Ae^t+1)^2 - (Ae^t-1)^2}{(Ae^t+1)^2}$$

$$= \frac{4Ae^t}{(Ae^t+1)^2} = 2y'(t)$$

so $y'(t) = \frac{1}{2}(1-y^2)$ for all A

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③ Let $f(x,y) = x^2 - x^4 + y^2$ find crit pts

$$\frac{\partial f}{\partial x} = 2x - 4x^3 \; ; \; \frac{\partial f}{\partial y} = 2y$$

Crit pts over $\begin{cases} 2x - 4x^3 = 0 \\ 2y = 0 \end{cases}$

(Graph: $z = f(x,y)$) | always $y=0$, $x \in \{\pm\frac{1}{\sqrt{2}}, 0\}$

Crit pts over $(0,0)$, $\left(\frac{1}{\sqrt{2}}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, 0\right)$

at $(0,0,0)$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{4}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{4}\right)$

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② Find max & min of $f(x,y) = -x^2 + 8y$ in $R = \{x^2+y^2 \leq 25\}$.

Since f is cts, R closed and bounded, f has max, min. the must occur at boundary or at critical pts (or at singular pts)

$\frac{\partial f}{\partial x} = f_x = -2x$; $\frac{\partial f}{\partial y} = 8 \neq 0$ so no critical pts

$\Rightarrow$ max, min are on boundary.

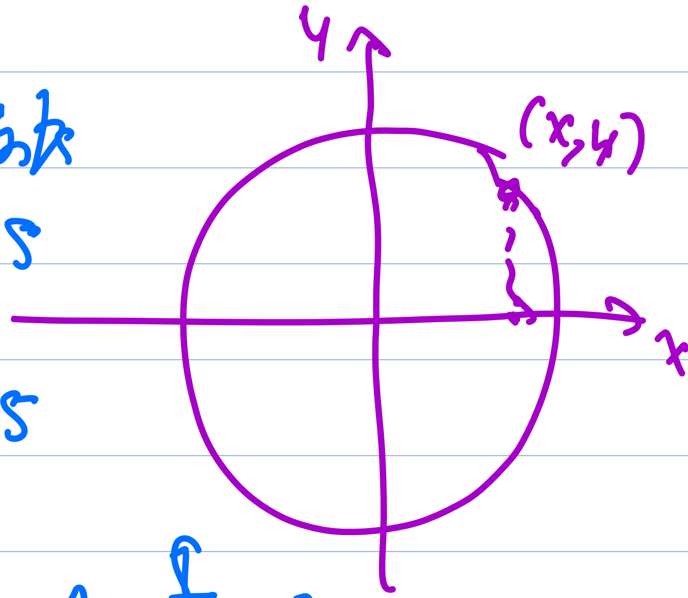
Want max & min of $-x^2 + 8y$ on curve $x^2+y^2=25$

Along the curve: $2x + 2y\frac{dy}{dx} = 0$

so $\frac{dy}{dx} = -\frac{x}{y}$

so $\frac{df}{dx} = -2x + 8\frac{dy}{dx} = -2x - 8\frac{x}{y}$

On boundary: no endpoints
Singularity at $y=0$, $x=\pm 5$



Crit pt if $x=0$, $y=\pm 5$

If $x \neq 0$, crit pt at $-2-\frac{8}{y}=0$

so $\frac{y}{8} = -\frac{1}{2}$, $y=-4$, $x=\pm 3$

$f(\pm 5, 0) = -25$, $f(0, \pm 5) = \pm 40$

$f(\pm 3, -4) = -9-32 = -41$

so max of f is 40, obtained at $(0,5)$
min is -41 " " $(\pm 3, -4)$

Or: <u>On boundary</u> $x^2 = 25 - y^2$

$$f(x,y) = -x^2 + 8y = -25 + y^2 + 8y$$
$$= -41 + (y+4)^2$$

$\Rightarrow$ min is -41 where $y=-4$, max where $y=5$

$P = -41 + 9^2 = 40$

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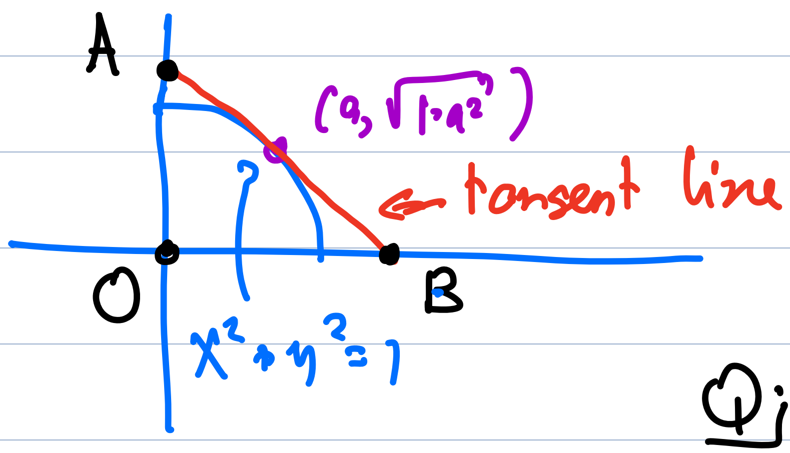
⑤ MT II optimization



AB : pts where tangent line meets axes

Q: minimize area of ΔABO.

Solution: along curve: $2x + 2y\frac{dy}{dx} = 0$

so $\frac{dy}{dx} = -\frac{x}{y}$

So slope of tangent line is $-\frac{a}{\sqrt{1-a^2}}$.

So tangent line is $Y = -\frac{a}{\sqrt{1-a^2}}(X-a) + \sqrt{1-a^2}$.

(1) At $X=0$, $Y = \frac{a^2}{\sqrt{1-a^2}} + \sqrt{1-a^2} = \frac{a^2+(1-a^2)}{\sqrt{1-a^2}} = \frac{1}{\sqrt{1-a^2}}$

So $|AO| = \frac{1}{\sqrt{1-a^2}}$.

(2) At $Y=0$, $\frac{a}{\sqrt{1-a^2}}(X-a) = \sqrt{1-a^2}$

So $aX - a^2 = 1 - a^2$ so $X = \frac{1}{a}$.

So $|BO| = \frac{1}{a}$.

(5) So area of ΔABO is $g(a) = \frac{1}{2a\sqrt{1-a^2}}$

arg min of $\frac{1}{2a\sqrt{1-a^2}}$ = arg max of $a\sqrt{1-a^2}$

= arg max of $a^2(1-a^2)$

let $b = a^2$ want max of $b(1-b)$
for $0 \leq b \leq 1$

at $b = 0, 1$ get 0

$\frac{d(b(1-b))}{db} = 1 - b - b = 1 - 2b$

so crit pt over $b = \frac{1}{2}$

value is $\frac{1}{4}$. so so max at $b = \frac{1}{2}$

so " " $a = \frac{1}{\sqrt{2}}$

$$\boxed{\begin{aligned} b(1-b) &= b - b^2 \\ &= -(b^2 - b) \\ &= -(b^2 - b + \tfrac{1}{4}) + \tfrac{1}{4} \\ &= \tfrac{1}{4} - (b - \tfrac{1}{2})^2 \end{aligned}}$$

So smallest area where $a = \frac{1}{\sqrt{2}}$.

so area is: $\dfrac{1}{2\cdot\frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} = \dfrac{1}{\sqrt{2}\sqrt{1-\frac{1}{2}}} = 1.$

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⑥

$$\frac{d}{d\theta}\tan\theta = 1+\tan^2\theta$$

$$\frac{d}{dt}(\tan\theta) = \frac{d\theta}{dt}\cdot(1+\tan^2\theta) = 2\frac{d\theta}{dt}$$

$x = 100\text{ m}$

$\tan\theta = 1$

$$\frac{d}{dt}(\tan\theta) = \frac{1}{100}\frac{dx}{dt}$$

$$\text{so } \frac{d\theta}{dt} = \frac{1}{200}\frac{dx}{dt} = \frac{3\text{ m}}{200\text{ sec}\cdot\text{m}}$$

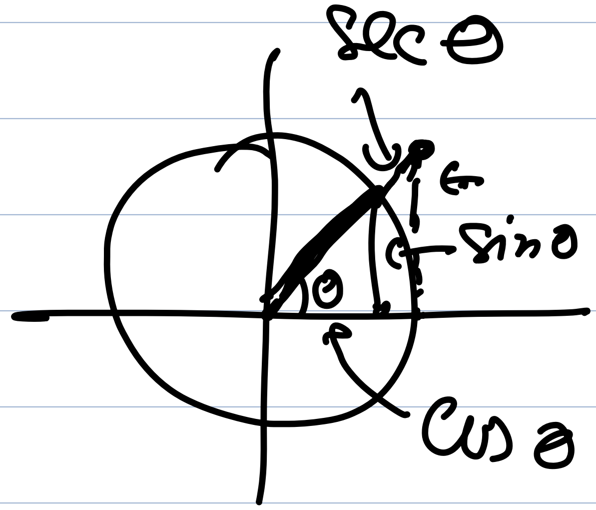
$$= \frac{3}{200}\frac{\text{rad}}{\text{sec}}$$

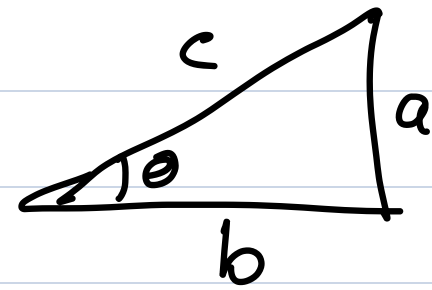
$\tan\theta = 1$ so $\theta = \frac{\pi}{4}$ so $\cos\theta = \left(\frac{1}{\sqrt{2}}\right)$

so $\sec^2\theta = 2.$

$\theta = \arctan\left(\frac{x}{100}\right)$ so $\dfrac{d\theta}{dt} = \dfrac{1}{1+\left(\frac{x}{100}\right)^2}\cdot\dfrac{1}{100}\cdot\dfrac{dx}{dt}$

(7)

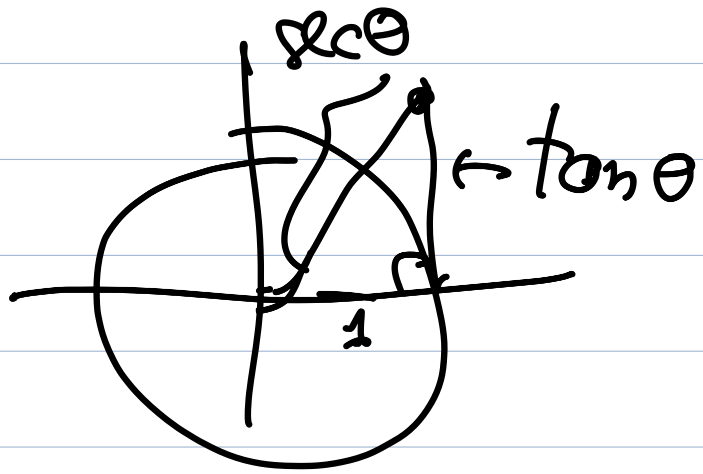




$\sin\theta = \frac{a}{c}$

$\operatorname{cosec}\theta = \frac{c}{a}$

$= \frac{1}{\sin\theta}$



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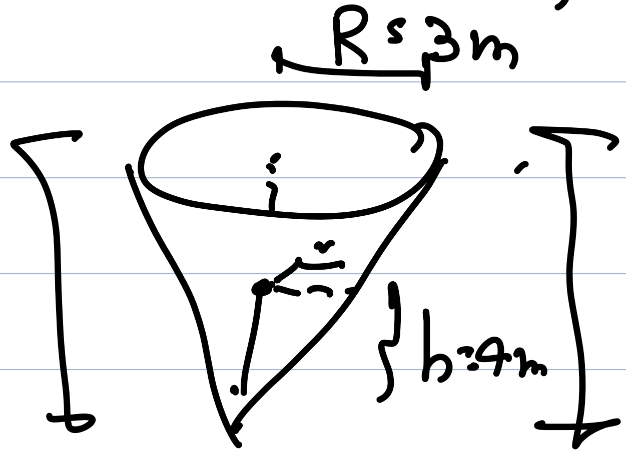
(8) Water fills an inverted cone.

It is leaking at rate of $3\,m^3/hr$ when $h = 4m$



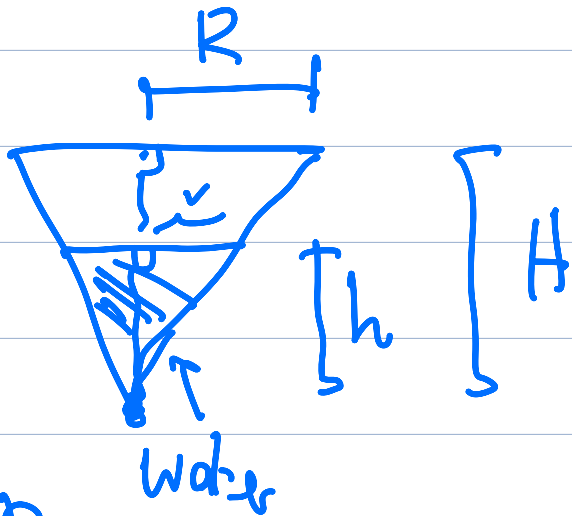
$H = Dm$

Want: $\frac{dh}{dt}$

Volume of water.

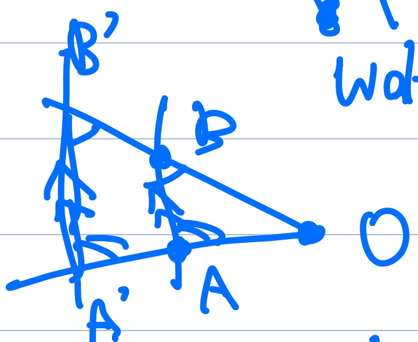
$$V = \frac{1}{3}\pi r^2 h$$

Vertical cross-section through tank.



from similar triangles:

$$\frac{r}{h} = \frac{R}{H}$$



So $r = \frac{R}{H} \cdot h$

So $V = \frac{1}{3}\pi \left(\frac{R}{H}\right)^2 h^3$

So $\frac{dV}{dt} = \pi \left(\frac{R}{H}\right)^2 h^2 \frac{dh}{dt}$

So $\frac{dh}{dt} = -\frac{3}{\pi} \cdot \left(\frac{8}{3}\right)^2 \cdot \frac{1}{4^2} = -\frac{4}{3\pi} \frac{m}{hr}$