Math 100:V02 – WORKSHEET 18
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

Definition. We say the point (x_0, y_0) is a *critical point* for the function $f = f(x, y)$ if f is defined in a neighbourhood of the point and

$$\begin{cases} \frac{\partial f}{\partial x}(x_0, y_0) = 0 \\ \frac{\partial f}{\partial y}(x_0, y_0) = 0 \end{cases}.$$

(1) ⋆How many critical points does $f(x, y) = x^2 - x^4 + y^2$ have?

(2) ⋆Find the critical points of $f(x, y) = x^2 - x^4 + xy + y^2$.

(3) (MATH 105 Final, 2013) ⋆ Find the critical points of $f(x, y) = xye^{-2x-y}$.

Date: 9/4/2023, Worksheet by Lior Silberman.

(4) **WARNING: in general checking along the axes only is not enough to determine if a point is a local minimum or maximum. For more on this look up the multivariable second derivative test in the reference book.**

(a) $\star\star$ Let $f(x, y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

(b) (MATH 105 Final, 2017) $\star\star$ Let $f(x, y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

(5) $\star$ Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.

2. Optimization

Fact. *The maximum and minimum of a function must (if they exist) occur either at (1) a critical point; (2) a singular point; (3) the boundary.*

(6) ⋆⋆Find the minimum of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$:

(a) on the rectangle $0 \le x \le 2$, $-1 \le y \le 1$.

(b) on the rectangle $2 \le x \le 3$, $-1 \le y \le 1$.

(7) Find the maximum of $(7x + 3y + 2y^2)e^{-x-y}$ for $x \ge 0$, $y \ge 0$,

(8) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C = 3t^2 + \sqrt{t} \cdot q$. At price p the company can sell $q = \frac{t-p}{3}$ widgets.

(a) Write an expression for the profit function $f(q, t)$.

(b) How many widgets of what quality should the company make to maximize profits?

(9) Find the maximum and minimum values of $f(x, y) = -x^2 + 8y$ in the disc $R = \left\{x^2 + y^2 \leq 25\right\}$.

(10) (MATH 105 final, 2015) Find the maximum and minimum values of $f(x, y) = (x-1)^2 + (y+1)^2$ in the disc $R = \left\{x^2 + y^2 \leq 4\right\}$.

(11) (The inequality of the means) We calculate the maximum of $f(x,y,z) = xyz$ on the domain $x+y+z = 1$, $x, y, z \geq 0$.

(a) Explain why it's enough to find the maximum of $g(x,y) = xy(1-x-y)$ on the domain $x \geq 0, y \geq 0$, $x+y \leq 1$.

(b) Find the critical points of g in the interior of the domain, and the values of g at those points.

(c) What is the boundary of the domain of g? What is the maximum there?

(d) What is the maximum of g?

(e) Show that for all $X, Y, Z \geq 0$ we have $(XYZ)^{1/3} \leq \frac{X+Y+Z}{3}$ (the "inequality of the means"). Hint: define $x = \frac{X}{X+Y+Z}$, $y = \frac{Y}{X+Y+Z}$, $z = \frac{Z}{X+Y+Z}$ and apply the previous result.

3. Lagrange multipliers (MATH 100C)

Fact (Method of Lagrange Multipliers)**.** *Let $f(x,y)$ and $G(x,y)$ be two functions (the* objective function *and the* constraint*). Suppose that (x_0,y_0) is a local maximum or minimum of f* ***restricted to the curve*** *$G(x,y)=0$. Then there is a number λ (the "Lagrange multplier") so that the following equations are satisfied:*

$$\begin{cases} \frac{\partial f}{\partial x}(x_0,y_0) = \lambda \frac{\partial G}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial y}(x_0,y_0) = \lambda \frac{\partial G}{\partial y}(x_0,y_0) \\ G(x_0,y_0) = 0 \end{cases}.$$

(11) (MATH 105 final, 2017) Use the mConstrained optimizationethod of Lagrange Multipliers to find the maximum value of the utility function $U = f(x,y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x,y) = 50x + 100y - 500,000 = 0$, where $x \geq 0$ and $y \geq 0$.

(12) Labour-Leisure model: a person can choose to spend L hours a day not working ("leisure"), working $24 - L$ hours with way w. Suppose their fixed income is V dollars per day. Their consumption of goods is them $C = w(24 - L) + V$, equivalenly $C + wL = 24w + V$ (here C, L are variables while w, V are constants). If their utility function is $U = U(C, L)$ find a system of equations for maximum utility.