Math 538, Lecture 11, 14/2/2024

Last time: Extended non-arch valuations to algebraic extensions via:
\[ |x|_L = |N_f^L x|_F^n \]

(if \( F \) complete)
\[ n = \left\lceil \frac{1}{e} \right\rceil. \]

Ex: remove completeness hypothesis.

Today: (1) degression
   (2) Ramification

(2) pf of extension relied on bounding \( |1+x|_L \)
for \( x \in \mathbb{Z} \) st. \( |x|_L \leq 1 \) (\( 1.L \) defined as above)

we get the bound \( |1+x|_L \leq 2 \) if \( |x|_L \leq 1 \)

\[ |\alpha + \beta|_L \leq \max \{ |\alpha|_L, |\beta|_L \}. \]

Enough to have \( |1+x|_L \) bounded. Better to define absolute values via:
(1) \( |xy| = |x| |y| \)
(2) \( |x| = 0 \) if \( x = 0 \) ; for some \( C \).
\[
( C = \sup \{ |1 + x| : |x| \leq 1 \} )
\]

**Observations:**
1. Strong enough to give usual theory of convergence.
   
   \[ \text{Ex: if } x_n \to x, \quad y_n \to y, \quad \text{then} \]
   \[ (|x_n + y_n| - |x+y|) \leq C \cdot \max \{ |x_n - x|, |y_n - y| \} \]

   so still get topological field.

2. Still true: either \( |l| \) bounded on \( \mathbb{D} \), [then \( C = 1 \)], or \( |l| \) unbounded, then \( |l_n| \geq 1 \) for all \( n > 1 \).

3. Now \( |l|^\lambda \) is an absolute value for \( \lambda > 0 \)

   \[ |l|, |l|^\lambda \text{ define same topology iff } |l|, |l|^\lambda \text{ are equivalent}. \]

   (In particular, natural on \( \mathbb{C} \) to take \( |x+iy| = x^2 + y^2 \).)
Example:

**Theorem:** Let \( K \) be a non-discrete locally compact field ("local field"). Then \( K \) is isomorphic to one of the following:

1. A finite extension of \( \mathbb{R} \)
2. A finite extension of \( \mathbb{Q}_p \), \( p \) prime
3. \( F_q((t)) \), \( q \) prime power

**Proof:** (Sketch) Let \( \mu \) be the Haar measure of \((K,+).\) For any \( a \in K^*\), \( x \mapsto ax \) is an automorphism, so \( \mu(aE) = |a| \cdot \mu(E) \) for all \( E \subset K \), for some number \( |a| \in \mathbb{R}_+ \) (also true if \( a = 0 \), with \( |0| = 0 \)).

Clearly, \( |ab| = |a| \cdot |b| \), \( |a| = \infty \) if \( a \neq 0 \).

Need to show \( |1| \) is cts, \( \exists \) \( \delta > 0 \) such that \( |x| < 1 \Rightarrow |1+x| < 1 + \delta \), \( x > 0 \).

\[ C = \sup \{ |1 + x| : |x| < 1 \} < \infty \]

\( \Rightarrow |1| \) is an absolute value, local compactness \( \Rightarrow K \) complete.
in char \(0, K \subset K, \) Ostrowsky: \(IR \) or \(\mathbb{Q}_p \subset K\)

local compactness \(\Rightarrow \sum k: \frac{E}{k} < \infty\)

In char \(p\) fields of constants = residue field

Details: see Weil, "Basic Number Theory.

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**Ramification**

Fix \(K\) complete wrt non-discrete non-arch absolute value \(1.1\), equip every algebraic extension with the unique extension of \(1.1\)

\(\mathcal{O}_K\subset K\) valuation ring, \(\mathfrak{p}\subset \mathcal{O}_K\) max ideal

\(K = \mathcal{O}_K/\mathfrak{p}\) residue field

write \(v = -\log 1.1\), a valuation on \(K\).

For an algebraic extension \(L/K\), write \(n=[L:K]\),

\(L_2 = \{x \in L : |x| \leq 1\}, \quad \mathfrak{p}_L = \{x \in L : |x| < 1\}, \quad \lambda = \mathcal{O}_K/\mathfrak{p}_L.\)
Def: The ramification index is 

\[ e = e(L/k) = \left[ v(L^x) : v(k^x) \right] = \left[ L^x : k^x \right] \]

The residue degree is \( f = f(L/k) = [\lambda : k] \).

Example: \( \mathbb{Q}_2(\sqrt{2}) : V_2(\sqrt{2}) = \frac{1}{2} \)

\( V_2(\mathbb{Q}_2(\sqrt{2})) = \frac{1}{2} \mathbb{Z} \), \( V_2(\mathbb{Q}_2) = \mathbb{Z} \)

\( e(\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2) = 2 \)

Cf. in \( \mathbb{Q}(\sqrt{2}) \), \( V = \mathbb{Z}[\sqrt{2}] \), \( (2) = (\sqrt{2})^2 \)

(in general, \( L/k \) fields, \( p \in \mathcal{O}_L \) prime,

\[ p\mathcal{O}_L = \prod_{i} \mathfrak{p}_i^{e_i} \]

See: completing \( K \) at \( p \), \( \mathcal{L} \) at \( \mathfrak{P}_i \); set ramification index \( e_i \) )

In relevant situation \( \mathcal{L}, K \) complete, \( p\mathcal{O}_L = p_L^e \)
Prop: \( n \geq e_0 \), if \( 1K^* | c \mathbb{B}_0^* \) is discrete
have equality.

Proof: Let \( 2w_i \choose i = c U_2 \) project to \( 2 \times \beta \)-basis \( A \).

Let \( \sum_j \prod_i c U_2 \) be \( 2 \times \beta \)-basis one coset representatives for \( 1K^* / 1K^* \).

Suppose \( \sum_{ij} x_{ij} w_i \prod_j = 0 \) for some \( x_{ij} \in K \)

Let \( s_j = \sum_i x_{ij} w_i \), so \( \sum_j s_j \prod_j = 0 \).

Fix \( j \), if not all \( x_{ij} = 0 \), define \( s'_j = \alpha_j \sum_i x_{ij} w_i \),
\( \alpha_j \in K^* \)
where \( \alpha_j x_{ij} \in U_2 \), at least one in \( U_2^* \).

\( \alpha_j = \frac{1}{x_{ij}} \) if \(|x_{ij}| \) largest.

Then \( s'_j \in U_2 \), mod \( p_2 \), \( s'_j = \sum_i (\alpha_j x_{ij}) \bar{w}_i \to 0 \).

Since \( 2\bar{w}_i \beta c \alpha \) are a basis, at least one \( \alpha_j x_{ij} \)
is nonzero.
So $|s_j| = 1$, $|s_j| = |x_j| = \max_I |x_{ij}| \leq |k^x^i|$

$\Rightarrow \forall j \in \sum s_j \Pi_j$, all nonzero summands have distinct absolute values, so if some $s_j \neq 0$

$\left| \sum_j s_j \Pi_j \right| = \max_j |s_j \Pi_j| \neq 0 \Rightarrow$

so all $s_j = 0$. Now $\exists \omega_i$'s indep $\Pi_k$, so all $\omega_{ij} = 0$

Assume now $v(k^x) = \mathbb{Z}$, $v(z^x) = \frac{1}{2} \mathbb{Z}$

Take $\Pi'_i = \Pi^i$ to be a absolute value $\frac{1}{2}$

("uniformize" = generator of value group $\mathfrak{v} \subset L$)

Take $\Pi_j = \Pi_j^j$, want: $\exists \omega_i \Pi_j \subset L$ is a basis

For this let $M = \bigoplus U_k \omega_i \Pi_j^j \subset U_L$

let $N = \bigoplus U_k \omega_i$. $N$ surjects on $U_2 / \Pi^2 = \sum_k U_k / \Pi^j$\n
so $U_2 = N + \Pi U_L$
(If $\alpha \in O$, have $x_i \in O$ so $\alpha = \sum x_i \omega_i e_F$.  
\( \pi O \)

\( \pi O \)

(\( \rho \leq \pi O \) since

\( \{ x \in O : |x| < 1 \} = \{ x \in O : |x| < 1 \} \pi \)

\( = \{ x \in O : |x| \leq 1 \} = \pi O \))

So  
\( O = N + \pi O \)

\( = N + \pi N + \pi^2 O \)

\( = N + \pi N + \pi^2 N + \pi^3 O \)

\( \vdots \)

\( \omega \),

\( = \sum_{j=0}^{\infty} \pi^j N + \pi^j \omega O = N + \omega \omega O \)

\( \omega \) uniformise for \( K \)

So  
\( O = N + \omega (N + \omega O) = N + \omega^2 O \)

by induction,  
\( O = N + \omega^k O \) for all \( k \).
\[ M \text{ is dense in } U_2 \quad (\mathcal{U}^k U_2 \text{ is a basis for the topology at } 0) \]

But \( U_1 \) closed in \( K \), so \( M = U_2^{ef} \) is closed in its \( K \)-span = \( K^{ef} \).

So \( N \) closed in \( L \), so \( M = U_2 \),
so \( \exists w_i \) span \( L \).

(to get \( n = \sum e_i f_i \); for \( \forall K \) \( A \) fields,
need: \( V \) place of \( K \) corresp to \( \mathbb{P}_V \mathcal{U}_K \)

Then \( L \otimes_{K} \mathcal{K}_V = \bigoplus_{w \in \Pi L} L w \)

Remark: Argument did not use \( n < \infty \)

Prove: if \( V(K^X) \) discrete \( e, f < \infty \)

Then \( n = ef < \infty \)
What if $|k^x|, |\ell^y|$ non-discrete.

maybe $v(k^x) = \mathbb{Z}[\mathbb{N}] \cap \mathbb{R}$