Math 538, lecture 11, 14/2/2024

Last time: Extended non-arch valuations to algebraic extensions via: $|x|_L = |N^L_F x|_F^{1/n}$
(if F complete)

$n = [L:F]$.

Ex: remove completeness hypothesis

Today: (1) digression
(2) Ramification

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(1) pf of extension relied on bounding $|1+x|_L$ for $x \in L$ s.t. $|x|_L \leq 1$ ($|\cdot|_L$ defined as above)

we got the bound $|1+x|_L \leq 1$ if $|x|_L \leq 1$

$$\Rightarrow |\alpha + \beta|_L \leq \max\{|\alpha|_L, |\beta|_L\}$$

Enough to have $|1+x|_L$ bounded. Better to define absolute values via:

(1) $|xy| = |x|\,|y|$; (3) $|x+y| \leq C \max\{|x|, |y|\}$

(2) $|x| = 0$ iff $x = 0$; for some C.

$(C = \sup\{ |1+x| \,;\, |x| \le 1 \})$

<u>Observations:</u> (1) Strong enough to give usual theory of convergence.

E.g. if $x_n \to x$, $y_n \to y$, then

$$|(x_n+y_n) - (x+y)| \le C \cdot \max\{ |x_n - x|, |y_n - y| \} \xrightarrow[n\to\infty]{} 0$$

so still get topological field.

(2) Still true: either $|\cdot|$ bounded on $\mathbb{Z}$, [then $C=1$], or $|\cdot|$ unbounded, then $|n|>1$ for all $n>1$.

(3) Now $|\cdot|^\lambda$ is an absolute value for all $\lambda>0$

$|\cdot|_1$, $|\cdot|_2$ define same topology iff $|\cdot|_1 = |\cdot|_2^\lambda$.

(in particular, natural on $\mathbb{C}$ to take $|x+iy| = x^2+y^2$.

Example:

Theorem: Let K be a non-discrete locally compact field ("local field"). Then K is isomorphic to one of the following:

(1) A finite extension of $\mathbb{R}$

(2) A finite extension of $\mathbb{Q}_p$, p prime

(3) $\mathbb{F}_q((t))$, q prime power

Proof: (sketch) Let μ be the Haar measure of $(K,+)$. For any $a \in K^\times$, $x \mapsto ax$ is an aut. So $\mu(aE) = |a| \cdot \mu(E)$ for all $E \subset K$, for some number $|a| \in \mathbb{R}^\times_{>0}$ (also true if $a=0$, with $|0|=0$)

Clearly $|ab| = |a|\,|b|$, $|a| \neq 0$ if $a \neq 0$

Need to show $|\cdot|$ is cts, $\{x : |x| \leq 1\}$ is cpt.

$\Rightarrow$ $C = \sup\{|1+x| : |x| \leq 1\} < \infty$

$\Rightarrow$ $|\cdot|$ is an absolute value, local compactness $\Rightarrow$ K complete

in char 0, $\mathbb{Q} \subset K$, Ostrowsky: $\mathbb{R}$ or $\mathbb{Q}_p \subset K$
local compactness $\Rightarrow$ $[K : {\mathbb{R} \atop \mathbb{Q}_p}] < \infty$

In char p fields of constants = residue field $\blacksquare$

Details: see Weil, "Basic Number Theory".

Ramification

Fix K complete wrt non-discrete non-arch absolute value $|\cdot|$, equip every algebraic extension with the unique extension of $|\cdot|$

$\mathcal{O}_K \subset K$ valuation ring, $\mathfrak{p} \triangleleft \mathcal{O}_K$ max'l ideal
$k = \mathcal{O}_K/\mathfrak{p}$ residue field

write $v = -\log|\cdot|$, a valuation on K.

For an algebraic extension L/K, write $n = [L:K]$,
$\mathcal{O}_L = \{\alpha \in L : |\alpha| \leq 1\}$, $\mathfrak{p}_L = \{\alpha \in L : |\alpha| < 1\}$, $\lambda = \mathcal{O}_L/\mathfrak{p}_L$.

Def: The ramification index is

$$e = e(L/K) = [v(L^\times) : v(K^\times)] = [|L^\times| : |K^\times|]$$

The residue degree is $f = f(L/K) = [\lambda : \kappa]$.

Example: $\mathbb{Q}_2(\sqrt{2})$: $v_2(\sqrt{2}) = \frac{1}{2}$

$$v_2(\mathbb{Q}_2(\sqrt{2})) = \tfrac{1}{2}\mathbb{Z}, \quad v_2(\mathbb{Q}_2) = \mathbb{Z}$$

$$e(\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2) = 2$$

Cf. in $\mathbb{Q}(\sqrt{2})$, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$, $(2) = (\sqrt{2})^2$

(in general, L/K # fields, $\mathfrak{p} \triangleleft \mathcal{O}_K$ prime,

$$\mathfrak{p}\mathcal{O}_L = \prod_i \mathfrak{P}_i^{e_i}$$

See: completing K at $\mathfrak{p}$, L at $\mathfrak{P}_i$, get ramification index e_i)

In relevant situation L, K complete, $\mathfrak{p}\mathcal{O}_L = \mathfrak{p}_L^e$

Prop: $n \geq ef$, if $|K^\times| \subset \mathbb{R}^\times_{>0}$ is discrete have equality.

Proof: Let $\{w_i\}_{i=1}^f \subset \mathcal{O}_L$ project to a k-basis 4 λ.

Let $\{\pi_j\}_{j=0}^{e-1} \subset \mathcal{O}_L$ be s.t. $\{|\pi_j|\}_{j=0}^{e-1}$ are coset representatives for $|L^\times|/|K^\times|$.

Suppose $\sum_{i,j} x_{ij} w_i \pi_j = 0$ for some $x_{ij} \in K$

Let $s_j = \sum_i x_{ij} w_i$, so $\sum_j s_j \pi_j = 0$

Fix j, if not all $x_{ij} = 0$, define $s_j' = \alpha_j \sum_i x_{ij} w_i$

where $\alpha_j \in K^\times$, $\alpha_j x_{ij} \in \mathcal{O}_L$, at least one in $\mathcal{O}_L^\times$.

$\alpha_j = \frac{1}{x_{ij}}$ if $|x_{ij}|$ largest

Then $s_j' \in \mathcal{O}_L$, mod $\mathfrak{p}_L$, $\overline{s_j'} = \sum_i \overline{(\alpha_j x_{ij})} . \overline{w_i} \neq \overline{0}$

Since $\{\overline{w_i}\} \subset \lambda$ are a basis, at least one $\overline{\alpha_j x_{ij}}$ is nonzero

So $|s_j'|=1$, $|s_j| = |\alpha_j| = \max_i |x_{ij}| \in |K^\times|$

$\Rightarrow$ in $\sum_j s_j \pi_j$ all nonzero summands have distinct absolute values, so if some $s_j \neq 0$

$$\left|\sum_j s_j \pi_j\right| = \max_j |s_j \pi_j| \neq 0 \Rightarrow \Leftarrow$$

so all $s_j = 0$. Now $\{w_i\}$ indep $/K$, so all $x_{ij}=0$

Assume now $v(K^\times) = \mathbb{Z}$, $v(L^\times) = \frac{1}{e}\mathbb{Z}$

Take $\pi_1 = \pi$ to be of absolute value $\frac{1}{e}$.

("uniformizer" = generator of value group of L)

Take $\pi_j = \pi^j$, want: $\{w_i \pi^j\} \subset L$ is a basis

For this let $M = \bigoplus_{i,j} \mathcal{O}_K w_i \pi^j \subset \mathcal{O}_L$

let $N = \bigoplus_i \mathcal{O}_K w_i$. N surjects on $\lambda = \mathcal{O}_L/p_L = \mathcal{O}_L/\pi$

so $\mathcal{O}_L = N + \pi \mathcal{O}_L$

(if $\alpha \in \mathcal{O}_L$, have $x_i \in \mathcal{O}_K$ st $\alpha - \sum_i x_i \omega_i \in \mathfrak{p}_L = \pi\mathcal{O}_L$

i.e. $\alpha \in N + \pi\mathcal{O}_L$)

($\mathfrak{p}_L = \pi\mathcal{O}_L$ since

$$\{x \in L : |x| < 1\} = \{x \in L \mid |x| \leq |\pi|\}$$
$$= \{x \in L \mid |\tfrac{x}{\pi}| \leq 1\} = \pi\mathcal{O}_L \text{)}$$

so $\mathcal{O}_L = N + \pi\mathcal{O}_L = N + \pi(N + \pi\mathcal{O}_L)$

$$= N + \pi N + \pi^2\mathcal{O}_L$$
$$= N + \pi N + \pi^2 N + \pi^3\mathcal{O}_L$$
$$\vdots$$
$$= \sum_{j=0}^{e-1} \pi^j N + \pi^e\mathcal{O}_L = M + \varpi\mathcal{O}_L$$

ϖ = uniformiser for K

so $\mathcal{O}_L = M + \varpi(M + \varpi\mathcal{O}_L) = M + \varpi^2\mathcal{O}_L$

by induction, $\mathcal{O}_L = M + \varpi^k\mathcal{O}_L$ for all k

$\Rightarrow M$ is dense in $\mathcal{O}_L$ ($\varpi^k \mathcal{O}_L$ is a basis for the topology at 0)

But $\mathcal{O}_K$ closed in K, so $M \cong \mathcal{O}_K^{ef}$ is closed in its K-span $\cong K^{ef}$.

so M closed in L, so $M = \mathcal{O}_L$,
so $\{\omega_i \pi^j\}$ span L. ∎

(to get $n = \sum_i e_i f_i$ for L/K # fields,
need: v place of K corresp to $\mathfrak{p}_v \triangleleft \mathcal{O}_K$

Then $L \otimes_K K_v \cong \bigoplus_{\substack{w \in |L| \\ w|v}} L_w$)

Remarks: Argument did not use $n < \infty$

Proves: if $v(K^\times)$ discrete $e, f < \infty$
then $n = ef < \infty$.

What if $|K^\times|$, $|L^\times|$ non-discrete.

maybe $v(K^\times) = \mathbb{Z}[\sqrt{2}] \subset \mathbb{R}$