Last time: Ramification

K complete wrt non-trivial, non-arch ds value
K perfect, 1.1 discrete

Prop: $L/K$ finite totally ramified iff
$L = K(T)$ where $T$ uniformiser; min poly
is Eisenstein

Thm: $O_p$ has finitely many extensions
of any fixed degree.

Today: Implications for number fields

Lemma: Let $L/K$ be an extension of fields, 1.1,
on absolute value on $L$, trivial on $K$. Then $1.1_w$
is trivial on the algebraic closure of $K$ in $L$

pf: $L$ is non-arch, let $\alpha \in \mathbb{A}$ with $|\alpha|_w > 1$
let $f(\mathbb{K}_x)$ be its min poly (assumed $\alpha \in \mathbb{A}/\mathbb{K}$)
$$t(x) = \sum_{i=0}^{d-1} a_i x^i + x^d.$$ 

$$\forall i \ni a_i x^i \mid_w = 1 \alpha_i x^i < (\alpha)_w^d \text{ if } i < d$$

$$\Rightarrow 1 f(\alpha)_w = 1 \alpha d \Rightarrow \text{ contradiction}$$

Fix a finite extension $L/K$ of fields, $v$ place of $K$. ($1/v$ is an absolute value)

Goal: Extend $v$ to $L$

Can do this: say $L$ gets by roots of some $f$

Then splitting field of $f$ over $K_v$ will contain a copy of $L$, so $v$ extends to $L$.

**Lemma:** There is a bijection between

$$\{w \mid L \mid v\} \leftrightarrow \text{Hom}(L; \overline{K_v})/\text{Gal}(\overline{K_v}/K_v)$$

**Pf:** $v$ extends uniquely to $\overline{K_v}$, so extension is $\text{Gal}(K_v)$-injective. Get map

$$\text{Hom}(L; \overline{K_v})/\text{Gal}(\overline{K_v}/K_v) \rightarrow \{w \mid L \mid v\}$$

It is surjective, if $w|v$, $Lw$ is a finite extension.
$L \cdot K_v = L_w$ is a fid. $K_v$-sp.

$\Rightarrow$ closed in $L_w$, contains $L$, so $L_w = L \cdot K_v$

$\Rightarrow$ $L_w$ finite over $K_v$ so algebraic

$\Rightarrow$ have embedding $L_w \subset \overline{K_v}$

compatible with absolute values by uniqueness

For injectivity, suppose $L, L' \subset \overline{K_v}$ subfields finite over $K$, $\sigma : L \to L'$ isometric $K$-isom. Need to show $\sigma$ extends to an aut. of $\overline{K_v}$.

First, $\sigma$ extends to the topological closures of $L, L'$ (finite extensions of $K_v$). These a fields, the extension is a $K_v$-isom. Done by Galois theory.

Remarks: we implicitly assumed $K_v / K$ is Galois. Check what happens otherwise.

Cor: Suppose $L = K(x)$ with min poly $f(x) \in K[x]$.

Then places of $L$ above $v$ are in bijection with irreducible factors of $f$ in $K_v[x]$. 
Recall: if \( L_2 = L_k[G] \) saw primes of \( L_2 \) above \( p \in \mathfrak{P}_k \) are bijection with factors of \( f \) in \( L_k[G] \).

Get new proof of this fact, works even if \( p \) is ramified in \( L/k \).

Example: if \( L/k \), fields, \( \nu \in \mathfrak{P}_L \).

If \( \nu = \mathfrak{C} \) then \( L_w = \mathfrak{C} \) for all \( w|\nu \).

If \( L = K(a) \), \( a \) has \( n \) embeddings to \( \mathfrak{C} \).

If \( K \) is \( R \), \( f \) factors in \( \mathbb{R}\Sigma x J \) into \( r \) linear, \( s \) quadratic factors

Get \( r \) real places, \( s \) complex places of \( L \) lying over \( \nu \).

In particular if \( K = \mathbb{Q} \), see that \( L \) has \( r \) real, \( s \) complex places where \( r + 2s = n = [L:K] \).

(Aut_{ch}(G) = \{1, c^3 \} acts on embeddings \( L \to \mathfrak{C} \), orbits are of size 1, 2)
(2) $K = \mathbb{Q}, \ L = \mathbb{Q} \left( \sqrt[6]{2} \right), \ \text{min pol} \ x^2 - 2.$

Over $\mathbb{Q}_{\infty} = \mathbb{R}$, $x^2 - 2 = (x - \sqrt[6]{2})(x + \sqrt[6]{2})$
so have one real place, one complex place

$$\dim_{\mathbb{R}} (\mathbb{R} \otimes \mathbb{C}) = 2 = [L : \mathbb{Q}]$$

Over $\mathbb{Q}_2$, $x^2 - 2$ is Eisenstein hence irreducible (irred in $\mathbb{Q}_2[[x]]$) get unique place $w_2 | \mathfrak{p}_2$, extension is totally (but tamely) ramified

Over $\mathbb{Q}_3$, $x^2 - 2 = x^3 + 1 = (x + 1)^3 = (x - 2)^3$

$f(2) = 6 = 3 \cdot 2, \ f'(2) = 3 \cdot 2^2 \ \text{so} \ \left| \frac{f(2)}{f'(2)} \right|_3 > 1$

Hensel’s lemma does not apply. In fact $f(2) = 6, \ f(3) = 12 + 26 = 38 \equiv 6 \pmod{9}, \ f(-1) = -3 \equiv 6 \pmod{9}$

$f$ has no root mod 9, hence in $\mathbb{Z}_9$, hence in $\mathbb{Q}_3$, so $f$ is irreducible in $\mathbb{Q}_3$. 


Let $g(y) = f(y-1) = y^3 - 3y^2 + 3y - 3$ (min poly of $1 + \sqrt{2}$). Also Eisenstein, so irreducible over $\mathbb{Q}_3$, which is totally ramified, $1 + \sqrt{2}$ is a uniformiser.

$N_{\mathbb{Q}_3}(1 + \sqrt{2}) = 3$ so $(1 + \sqrt{2})$ is the prime ideal $3$.

Over $\mathbb{Q}_p$: mod 5, $f = (x-3)(x^2 + 3x + 4)$. 2nd factor is irreducible $f'(3) = 27 \equiv 2 \pmod{5}$.

So by Hensel's Lemma, $f \in \mathbb{Q}_p[x]$ with $b$, of deg 1, $b \in \mathbb{Q}_p$ is 2. irreducible.

⇒ two places over 5, one completion $\mathbb{Q}_5$ other completion $L_w$ is a quad extension of $\mathbb{Q}_5$, the unramified extension since $\overline{f}_2$ is irreducible.

Over $\mathbb{Q}_p$, $p \neq 5$, $f' = 3x^2$ red prime to $\overline{f}$ so any root of $\overline{f}$ mod $p$ lifts to $\mathbb{Q}_p$, if $\overline{f} = \overline{f}_1$ this lifts to $f = \overline{f}_1$.

3 places $\mathbb{Q}_p(\sqrt{2})$ over $p$ correspond to $\overline{f}_1$. 
All unram since \( F \) irreducible mod \( p \)

\[ \text{\( \Phi \) if \( p \equiv 1 \pmod{3} \), \( \mathbb{Z}/p \mathbb{Z} \) has cube roots of unity.} \]

\[ \Rightarrow \mathbb{Z}_p \cong \mathbb{Q}_p \text{ have cube roots of unity.} \quad \Rightarrow \mathbb{Q}_p \cong \mathbb{Q}(\sqrt[3]{3}) \]

\[ \Rightarrow \text{either } f \text{ irreducible or has 3 linear factors} \]

\[ p \text{ is inert in } L/\mathbb{Q} \quad p \text{ splits in } L/\mathbb{Q} \]

\[ f \text{ splits if } \exists \text{ root of } f \text{ in } \mathbb{Z}/p \mathbb{Z} \]

\[ \begin{align*}
\text{let } p &= \pi \pi \in \mathbb{Z}[\sqrt{3}] \text{, } f_{\mathbb{Q}/\mathbb{Q}}(\pi; p) = 1 \quad \text{efg} = 2, \ g = 2 \\
& \text{or } \mathbb{Q}(\sqrt[3]{3}) \text{ completed at } \pi \text{ is } \mathbb{Q}_p \text{, so } f = 1 \\
& \mathbb{Z}[\sqrt[3]{3}]/\pi \mathbb{Z}[\sqrt[3]{3}] = \mathbb{Z}/p \mathbb{Z}.
\end{align*} \]

Need to decide if \( (\frac{3}{\pi})_3 = 1 \). By cubic reciprocity this is \( \Rightarrow (\frac{\pi}{3})_3 \) (choose \( \pi = \pm 2 \pmod{3} \) in \( \mathbb{Z}[\sqrt[3]{3}] \))

\( 2 \) is prime in \( \mathbb{Z}[\sqrt[3]{3}] \), \( \mathbb{Z}[\sqrt[3]{3}]/2 \mathbb{Z}[\sqrt[3]{3}] \cong \mathbb{F}_4 \)

\( \text{Only cube there is } 1 \Rightarrow (\frac{3}{\pi})_3 = 1 \text{ if } \pi \equiv 1 \pmod{2} \)

\[ \text{write } p = a^2 + 3b^2 \quad (p = N\pi), \quad \text{condition on splitting in terms of } a, b \]
\( p = 2 \) (2) \((\mathbb{Z}/2\mathbb{Z})^\times = \mathbb{Z}/3\mathbb{Z}\) so no cube root of unity in \(\mathbb{Q}/\mathbb{Z}\) so in \(\mathbb{Q}\)

so either \( p \) inert (\( f \) irreducible) or \( f = f_1 f_2 \).

Sweedler - Rosen, A Classical Introduction to Modern Number Theory (GSM)

Return to general problem. \( L \) finite, \( v \) place of \( K \). Saw: completions of \( L \) lie in \( L \cdot K_v \).

Makes us automatically interested in the \( K_v \)-algebra

\[ K_v \otimes_k L \]

for each \( w | v \) get hom \( K_v \otimes_k L \rightarrow L_w \)

so get hom

\[ K_v \otimes_k K \rightarrow \bigoplus_{w | v} L_w \] (4)

(from embedding \( L_w \rightarrow K_v \) get map back)
Thm: If $L/K$ is separable, $(a)$ is an isom

**Pr:** Say, $L = K(x)$ with min poly $f(x) \in K[x]$.
Say, $f = \prod w f_w$ with $f_w \in K_v[x]$ irreducible (distinct since $f$ is separable). By CRT:

$$\prod_L \mathcal{O}_L = \prod_w \mathcal{O}_w \left( K_v[x] \setminus (f_w) \right) = K_v \otimes_{K[x]} \mathcal{O}_L$$

$$= K_v \otimes_{K} K[x] = K_v \otimes_{K} L.$$

**Cor:** If $L/K$ is separable,

$$\sum_{L \mid K} = \sum_{W \mid L} \mathcal{O}_L$$

$$\sum_{W \mid L} e(\mathfrak{p} W L) = e(\mathfrak{m} \mathfrak{n} L)$$

valuation is discrete

residue field perfect

---

$\text{SL}_2(\mathbb{Z}[\sqrt{-c}]) \subset \Gamma \subset \Gamma_{c}$

action totally discontinuous

quotient has finite volume

$\rightarrow$ Hilbert modular forms
$R_v$ top rings, $k_v \subset R_v$ compact subrings

$M_v$ $R_v$-modules

$\otimes_v M_v$

This relates to the question on restricted tensor products in the adele supplement

Need: $\xi_v \in M_v$ st. $k_v \subset \xi_v$ for almost all $v$

$\otimes_v \otimes_v \xi_v = \otimes_v M_v$

$\exists \xi_v$ $G_v$ group, $k_v$ subsys $M_v$ reps

$\xi_v \in M_v$ $k_v$-fixed

$\otimes_v M_v$ is a repn of $\prod_v G_v$

Tems (Flath) $F$ #field, $G/F$ olg.

Every fixed adn repn of $G(A) = \prod (G(k_v):G(k_v))$

is of form $\otimes_v \eta_v$ at almost all places $v$