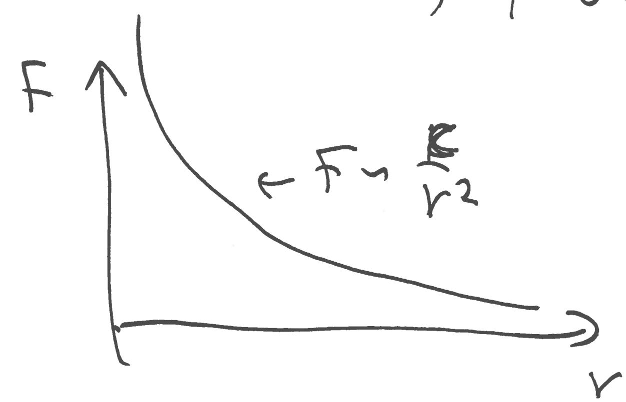
1. Asympototics (4/9/2024)

Today's Goals.

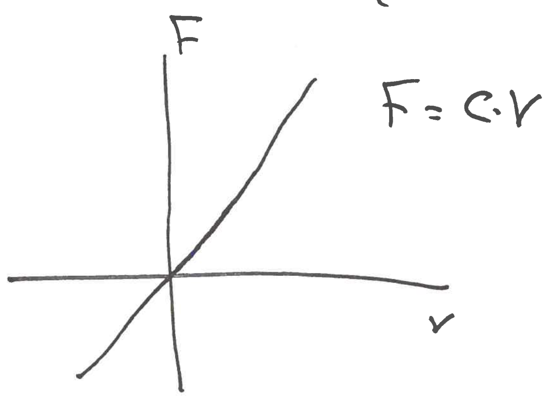
(1) Welcome!
(2) Power laws, exponentials, and their asymptotics
(3) Asymptotics of sums
(4) Asymptotics of expressions

Asymptotics: qualitative understanding of expressions

Inverse square law:



Hooke's law:



__Power law__: expression $A \cdot x^n$ (as function of x)

(call n __index__)

__Exponential__: expression $C \cdot b^x$ (as function of x)

(call b __base__)

$b > 0$

Math 100A – WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. The ladder of functions

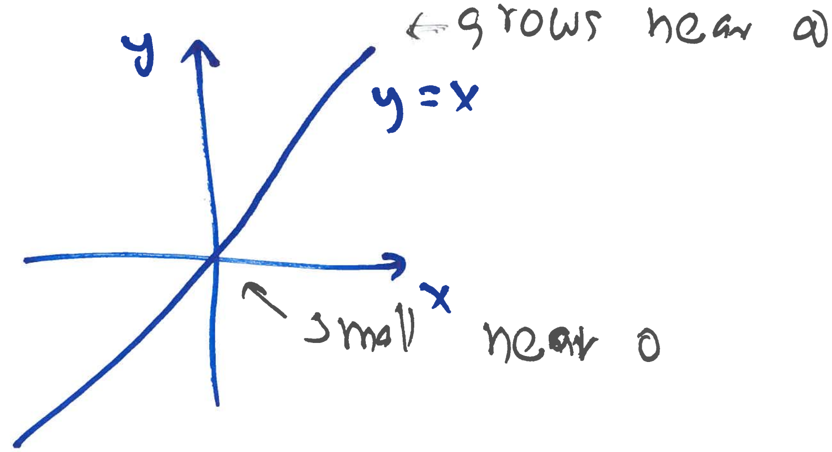
(1) Classify the following functions into *power laws / power functions* and *exponentials*: x^3, πx^{102}, e^{2x}, $c\sqrt{x}$, $-\frac{8}{x}$, 7^x, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, x^e, π^x, $\frac{A}{x^b}$.

Power laws: x^3, πx^{102}, $c\sqrt{x} = c \cdot x^{\frac{1}{2}}$, $-\frac{8}{x}$, $9 \cdot x^{-7/2}$, x^e, Ax^{-b}

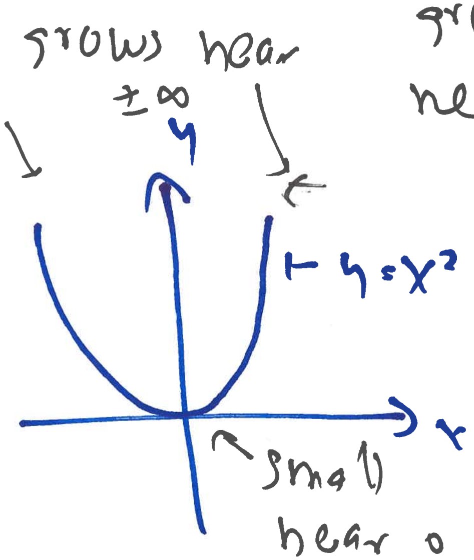
Exponentials: $e^{2x} = (e^2)^x$, 7^x, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \left(\frac{1}{2}\right)^x$, π^x,

Date: 4/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

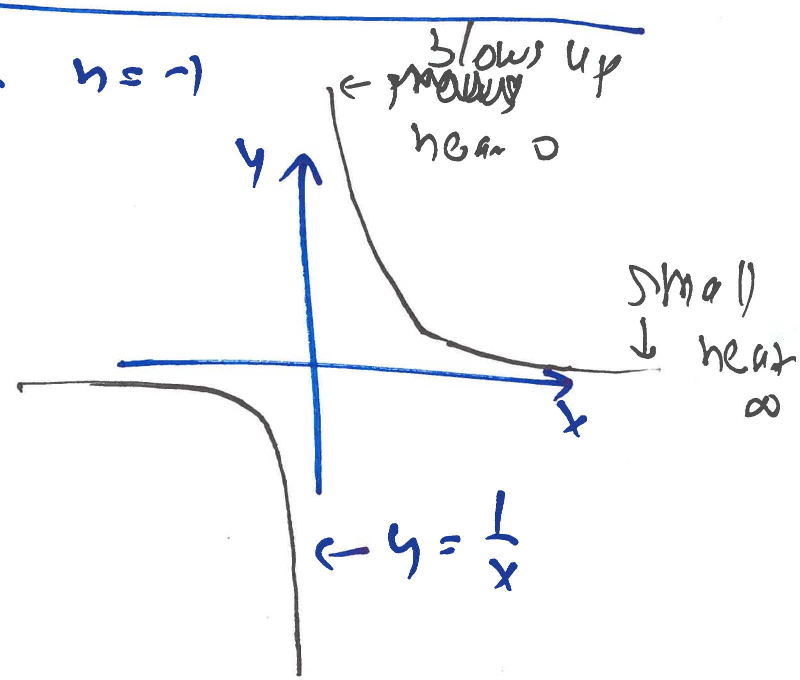
Power laws: if $n=1$

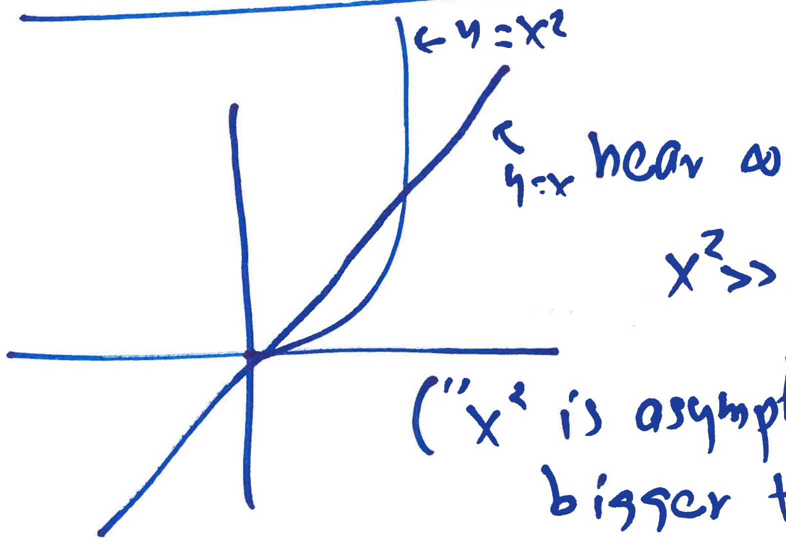


if $n=2$



if $n=-1$



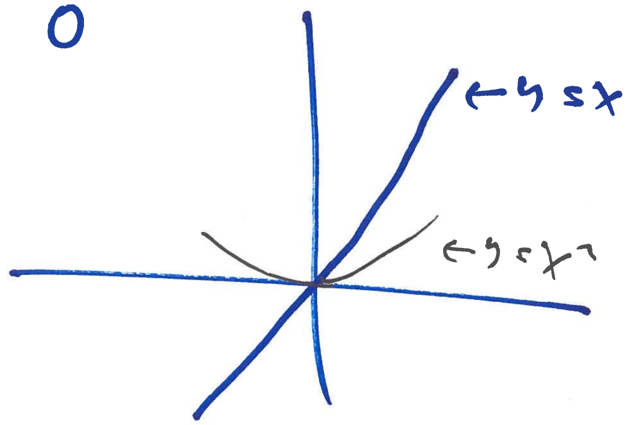


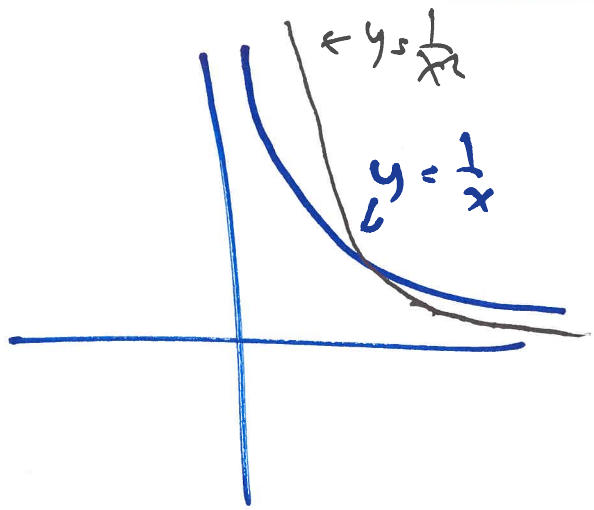
$x^2 >> x$

("x^2 is asymptotically bigger than x" as $x \to \infty$)

near 0

$x^2 << x$





as $x \to \infty$

$$\frac{1}{x} >> \frac{1}{x^2}$$

as $x \to 0^+$

$$\frac{1}{x} << \frac{1}{x^2}$$

("$\frac{1}{x}$ is a big number, $(\frac{1}{x})^2$ is even bigger.

Recap

__Idea:__ can compare functions "eventually" (say "__asymptotically__")

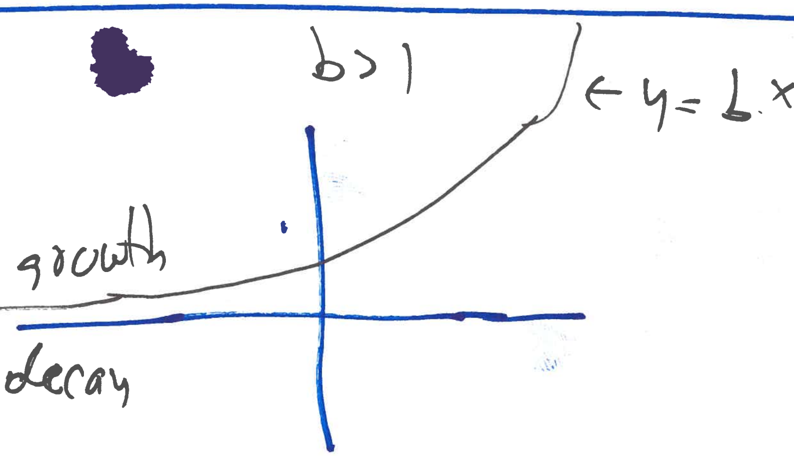
Have "$f >> g$ as $x \to \infty$": means as $x \to \infty$, __eventually__, f will be much greater.

Similarly look at asymptotics as $x \to 0$

or as $x \to 5$

⋮

__Add exponentials.__



as $x \to \infty$: exponential growth

as $x \to -\infty$: exponential decay

as $x \to 0$, $b^x \to 1$

(2) Order the following functions from small to large *asymptotically as* $x \to \infty$:

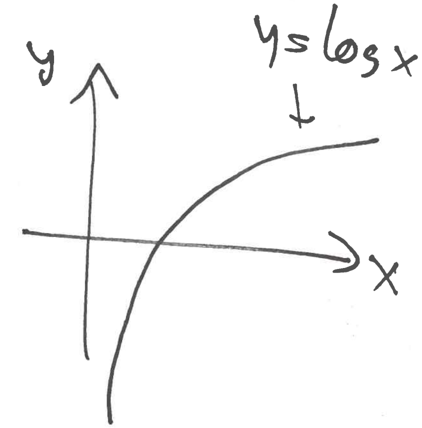
(a) 1, $\sqrt{x}$, $x^{-1/2}$, $x^{1/3}$, e^x, $x^{-1/3}$, $10^6 x^{2024}$, e^{-x}, e^{x^2}, $\frac{2024}{x^{100}}$, 5^x, x.

$\frac{1}{e^x} << \frac{2024}{x^{100}} << \frac{1}{\sqrt{x}} << \frac{1}{\sqrt[3]{x}} << 1 << x^{1/3} << x^{1/2} << x << 10^6 x^{2024} << e^x << 5^x$

$<< e^{x^2}$

(aside: $e^{x^2} = e^{x \cdot x} = (e^x)^x$

($(\log x)^{\log x}$ growing)



(b) Extra: add in $\log x$, $e^{\sqrt{x}}$, $(\log x)^2$, $\log \log x$, $\frac{1}{\log x}$.

logarithm to natural base

slower than all power laws

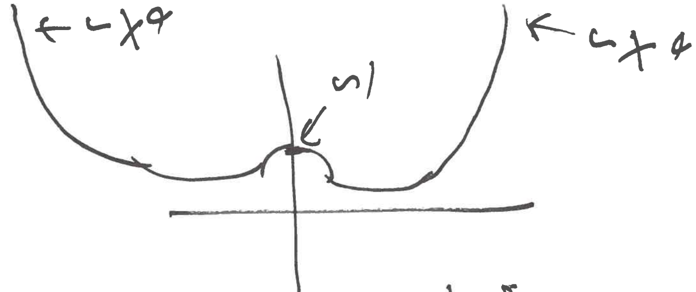
(3) How does the each expression behave when x is large? small? what is x is large but negative? Sketch a plot

(a) $1 - x^2 + x^4$ ("Mexican hat potential")

As $x \to \infty$, $1 - x^2 + x^4 \sim x^4$ ("the expression is asymptotic to x^4")

As $x \to 0$, $1 - x^2 + x^4 \sim 1$ (1 dominates x^2, x^4 as $x \to 0$)

as $x \to 0$, $1 - x^2 + x^4 < 1$
since $x^2 \gg x^4$

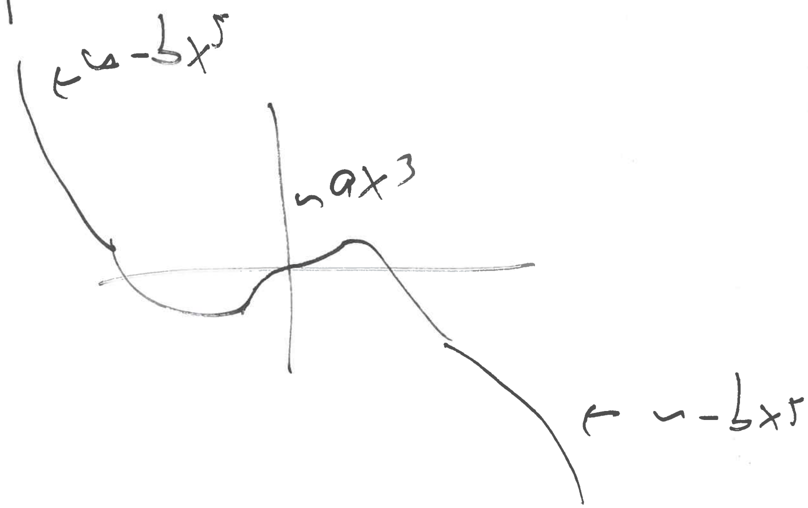


(b) $ax^3 - bx^5$ $(a, b > 0)$

As $x \to \infty$, $ax^3 - bx^5 \sim -bx^5$

$x \to -\infty$ $ax^3 - bx^5 \sim -bx^5$

As $x \to 0$ $ax^3 - bx^5 \sim ax^3$



Summary

In an asymptotic regime, (say $x \to \infty$, $x \to 0$, ..)

① Some ~~eg~~ functions will dominate others

② When adding effects, only dominant one matters

③ When multiplying, both matter.