2. Limits.Asymptotes, Continuity (11/9/2024)

Goals.

(1) Limits of functions
(2) Existence and nonexistence of limits: blowup
(3) Asymptotes
(4) Continuity

work as a group | must be typeset
recommend LaTeX (e.g. on overleaf.com or using LyX)

News: ∘ Calculus Common Room ∘ Group project 1.

Last Time. Asymptotics

if we have two functions $f(x), g(x)$ and a limit $x \to a$ (includes $x \to \infty$, $x \to -\infty$) can have $f(x) \ll g(x)$

as x nears a, g is relatively larger & larger than f

Can have $f \sim g$ means $\frac{f}{g} \to 1$

relatively f, g "same"

E.g. as $x \to \infty$, $x^4 \gg x^2$, $x^2+1 \sim x^2$ | as $x \to 0$, $x^4 \ll x^2$.

Math 100A – WORKSHEET 2
LIMITS. ASYMPTOTES, AND CONTIJUITY

(1) Review of asymptotics: analyze the expression $\frac{e^x + A\sin x}{e^x - x^2}$ as $x \to \infty$, $x \to 0$, $x \to -\infty$.

As $x \to \infty$, $A\sin x$ bounded (between $-A, A$) so $e^x \gg A\sin x$
also $e^x \gg x^2$

so $\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{e^x}{e^x} \sim 1$

As $x \to 0$, $e^x \sim 1$, $A\sin x, x^2 \to 0$
so $\frac{e^x + A\sin x}{e^x - x^2} \sim 1$

As $x \to -\infty$: $e^x \to 0$, $e^x - x^2 \sim (-x^2)$
but e^x, $A\sin x$ incomparable, so no simple asymptotics

1. LIMITS

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 5}(x^3 - x) = 125 - 5 = 120$

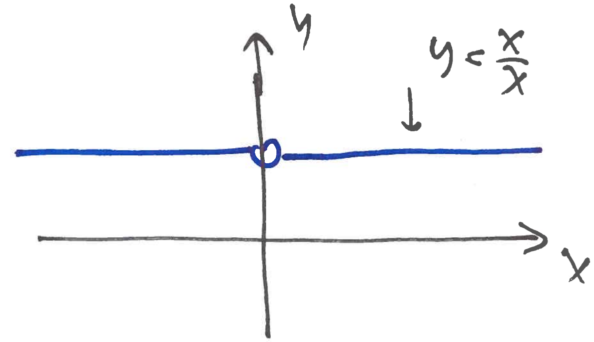
Date: 11/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Limits

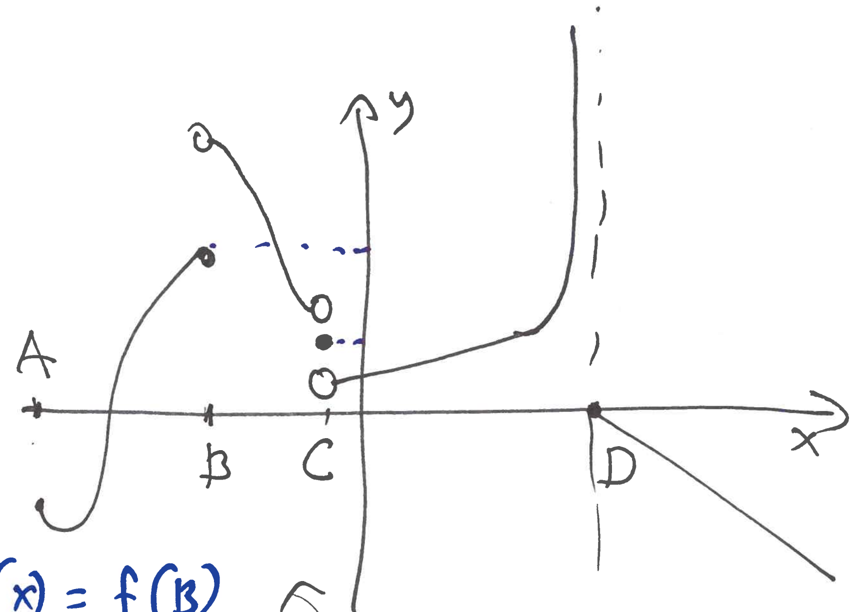
Say f is defined near a (on left and right)

Informally: $\lim\limits_{x\to a} f(x)$ is the value f "would like to have" at a.

Example: $\lim\limits_{x\to 0} \frac{x}{x} = \lim\limits_{x\to 0} 1 = 1$



Examples: a graph of some function



At B: $\lim\limits_{x\to B^-} f(x) = f(B)$ ← from left

$\lim\limits_{x\to B^+} f(x) =$ Value on top

disagree so $\lim\limits_{x\to B} f(x)$ Does not exist (no single value) "DNE"

Both B, C "jumps" in f

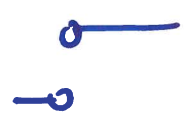
Promise: If f is defined by formula near a (including a) then $\lim_{x \to a} f(x) = f(a)$

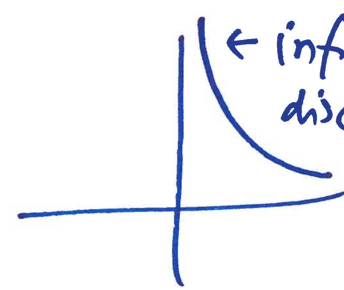
Back to graph: $\lim_{x \to D^+} f(x) = 0$

$\lim_{x \to D^-} f(x)$ DNE

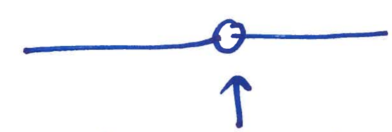
but $\lim_{x \to D^-} f(x) = \infty$ (in the extended sense)

If $\lim_{x \to a} f(x) = f(a)$ say f is continuous at a

jump $\rightarrow$ 



$\leftarrow$ infinite discontinuity



"removable" discontinuity

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

$$\lim_{x\to 1^-} f(x) \underset{x<1}{=} \lim_{x\to 1^-} \sqrt{x} = 1$$

$$\lim_{x\to 1^+} f(x) \underset{x>1}{=} \lim_{x\to 1^+} (2-x^2) = 1$$

$$\Rightarrow \lim_{x\to 1} f(x) = 1$$

(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

$$\lim_{x\to 1^-} f(x) = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4-x^2) = 3$$

$1 \neq 3$ so $\lim_{x\to 1} f(x)$ DNE (f jumps at 1) [note: $\dfrac{3-3}{3^2+3-12} = \dfrac{0}{0}$ undef]

(3) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

Since $x^2+x-12 = (x-3)(x+4)$ if $x \neq +3$ then $f(x) = \dfrac{1}{x+4}$

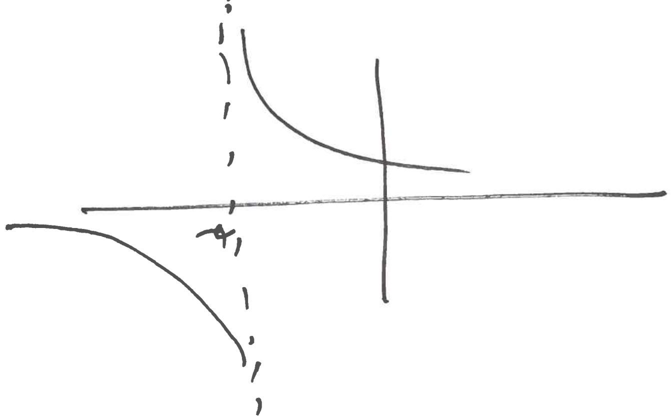
$\xrightarrow{x \to 3} \boxed{1/7}$

(b) What about $\lim_{x\to -4} f(x)$?

As $x \to -4$ $f(x) = \dfrac{1}{x+4}$ is blowing up $(x+4 \to 0)$

If $x > -4$, $x+4 > 0$, so $\dfrac{1}{x+4} > 0$ so $\lim\limits_{x\to -4^+} \dfrac{1}{x+4} = +\infty$

If $x < -4$, $x+4 < 0$ so $\dfrac{1}{x+4} < 0$ so $\lim\limits_{x\to -4^-} \dfrac{1}{x+4} = -\infty$



Lessons $\dfrac{1}{0}$ does not make sense

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

As $x \to -3^+$, $x+3 \to 0$, is positive
$x+2 \to -1$ negative

So $\frac{x+2}{x+3}$ blows up at -3,

when $x > -3$, $\frac{x+2}{x+3} < 0$

So $\lim_{x \to -3^+} \frac{x+2}{x+3} = -\infty$

Or: $x+2 \sim -1$
$x+3 \sim x+3$

So $\frac{x+2}{x+3} \sim \frac{-1}{x+3} \xrightarrow[x\to -3^+]{} -\infty$

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$

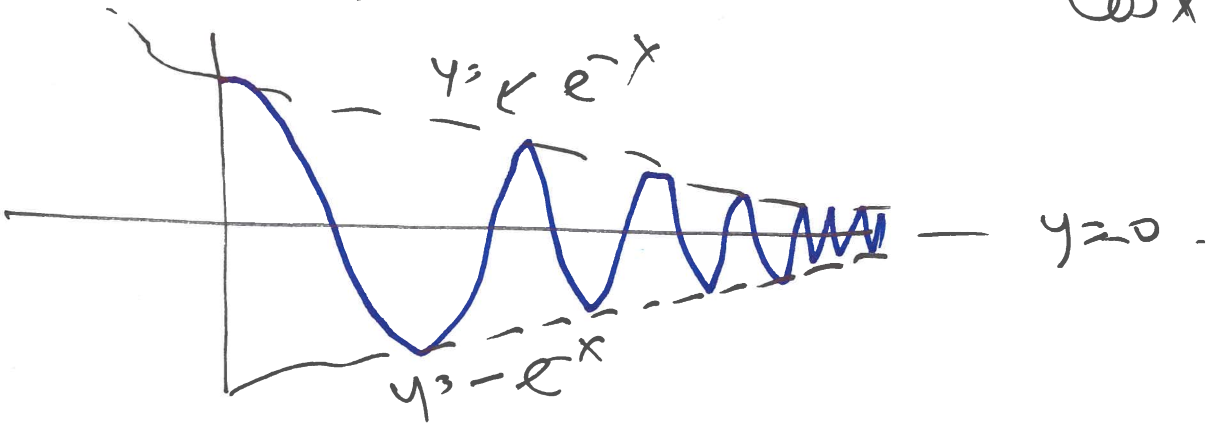
(d) $\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2}$

Asymptotes

If f blows up at $x=a$ (at least from one side of a) say f has the vertical asymptote $x=a$

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ exists say f has the horizontal asymptote $y=L$.

Example: $\lim_{x\to\infty} e^{-x}\cos x = 0$ (e^{-x} is decaying while $\cos x$ is bounded)



(c) $g(x) = \sqrt{\log x}$

(9) ("Gluing functions") In each problem find the value of the constant k such that the function is continuous.

(a) $f(x) = \begin{cases} \frac{x^3-2x^2}{x-2} & x \neq 2 \\ k & x = 2 \end{cases}$

(b) $g(x) = \begin{cases} 8 - kx & x < k \\ x^2 & x \geq k \end{cases}$

will be continuous if

$8-k^2 = k^2$

Value on left

Value on right