3. The Derivative (18/9/2024)

Goals.

(1) The derivative at a point
(2) Tangent lines & linear approximations
(3) The derivative as a function

Last Time. Limits

If f is defined near $x=a$, might have $\lim_{x\to a} f(x)=L$, which means: "as x gets closer to a, the values $f(x)$ get closer to L". [Extend to $L=\infty$, $L=-\infty$]

[Also consider limits as $x\to\infty$, $x\to-\infty$]

Continuity: $\lim_{x\to a} f(x) = f(a)$

The Derivative

- Suppose f is continuous at a. Then $f(x)$ is close to $f(a)$,
- so $f(x)-f(a)$ is decaying to 0 as $x \to a$

<u>Question:</u> <u>how</u> is $f(x)-f(a)$ decaying?

Measure this in terms of $h = x-a$ ← the "small parameter"

$\Updownarrow$

$x = a+h$

$\Leftarrow$ "wiggle" ~~x~~ a by moving to $a+h$

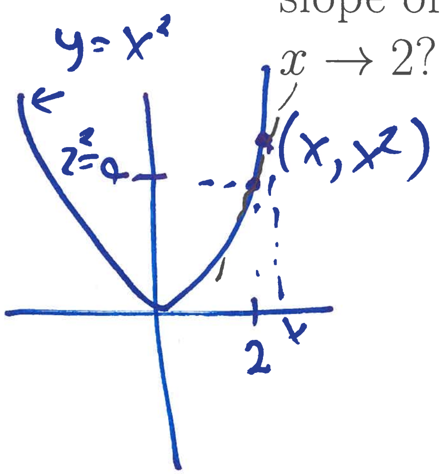
ask: "how does $f(x) = f(a+h)$ wiggle in response?"

Math 100A – WORKSHEET 3
THE DERIVATIVE

1. Three views of the derivative

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2, 4)$ is a point on the graph of $y = f(x)$.

(a) Let (x, x^2) be another point on the graph, close to $(2, 4)$. What is the limit of the slopes as $x \to 2$? What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?



know: if x is close to 2, x^2 is close to $2^2=4$.

line connecting two points on graph $((2,4),(x,x^2))$ is called a secant line

slope: $\dfrac{x^2-4}{x-2} \underset{x\neq 2}{=} x+2 \xrightarrow[x\to 2]{} 4.$

"the closer x is to 2, the more (x,x^2) is "on" line of slope 4 through $(2,4)$

Date: 18/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2+h)$ as $h \to 0$? What about $f(2+h) - f(2)$?

$$f(2+h) = (2+h)^2 \underset{h \to 0}{\sim} 4 + 4h + h^2 \sim 4$$

$$f(2+h) - f(2) = (4 + 4h + h^2) - 4 = 4h + h^2 \sim \boxed{4}h$$

"derivative of f at 2"

(c) What is $\lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h}$?

$$\lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} (4 + h) = \boxed{4}$$

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2, 4)$?

$$Y = 4(X - 2) + 4$$

Summary: Definitions of the derivative

Given function f, point a, typically $f(a+h)-f(h)$ has linear asymptotics. If $f(a+h)-f(a) \sim ch$ (as $h\to 0$) call c the "derivative of f at a". Write $f'(a)=c$

Equivalently, $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Equivalently, if $f(a+h) \approx f(a) + ch$ call c the derivative

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Call the fact $f(a+h) \approx f(a) + f'(a)h$
$\Leftrightarrow f(x) \approx f(a) + f'(a)(x-h)$
the "linear approximation"

Or: $\Delta k \approx 10\Delta T$
when $T = 20°C$

(2) An enzymatic reaction occurs at rate $k(T) = T(40 - T) + 10T$ where T is the temperature in degrees celsius. The current temperature of the solution is 20°C. Should we increase or decrease the temperature to increase the reaction rate?

$$k(T+h) - k(T) = (T+h)(40-T-h) + 10(T+h) - (T(40-T) + 10T)$$

$$= \cancel{T(40-T)} + h(40-T-h) - hT + \cancel{10T} + 10h - \cancel{T(40-T)} \cancel{-10T}$$

$$= 50h - 2Th - h^2 \sim (50-2T)h$$

$$= 10h - h^2 \qquad \sim 10h$$

(if we increase the temperature by h degrees, the reaction rate will change by $\approx 10h$)

So if $h>0$ $k(T+h)$ will be bigger (by about $10h$)

$\Rightarrow$ increase the temperature

[By def'n, the derivative of the reaction rate w.r.t. temp. is $50-2T$.

2. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a+h) \approx f(a) + f'(a)h$

(3) Use a definition of the derivative to find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

$f(a) = f(3) = 3^2$, $f(a+h) - f(a) = (3+h)^2 - 3^2 = 6h - h^2 \sim \boxed{6}h$ as $h \to 0$

So $f'(3) = 6$

(b) $f(x) = \frac{1}{x}$, any a.

$$f(a+h) - f(a) = \frac{1}{a+h} - \frac{1}{a} = \frac{a-(a+h)}{a(a+h)} = -\frac{h}{a(a+h)} \sim -\frac{h}{a\cdot a} = -\frac{h}{a^2}$$

$\uparrow$

$\left(-\frac{1}{a^2}\right)\cdot h$

So $f'(a) = -\frac{1}{a^2}$

(c) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

(4) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h) - \cos 5}{h}$, $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0}$

$\| \quad \frac{d}{dx}\Big|_{x=5} \cos(x)$

the derivative of $f(x) = \sin x$ at $x = 0$.

The derivative function

At each a we found a value $f'(a)$

The rule "map a to $f'(a)$" is a function. Denote this function f', $\frac{df}{dx}$, $\frac{d}{dx}f$, Df, $D_x f$, –

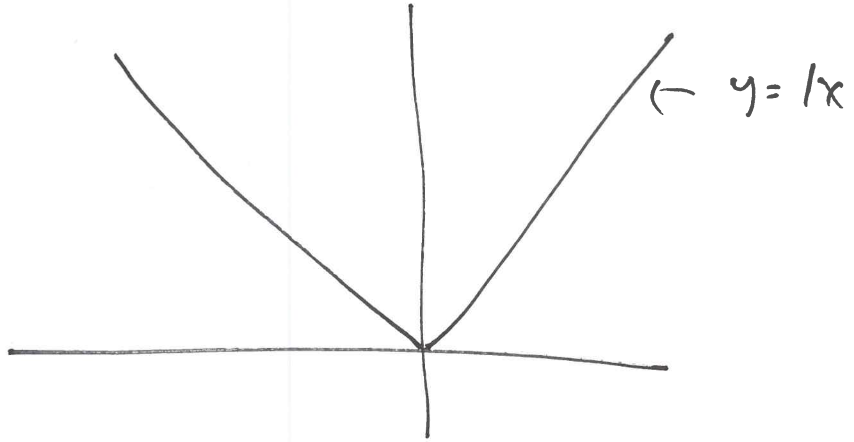
Call it "the derivative of f".

$f'(a)$ need not exist: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ need not exist.

Say f is **differentiable** at a, if $f'(a)$ exists

(the operation $\frac{d}{dx}$ is called <u>differentiation</u>)

Example, $f(x) = |x|$

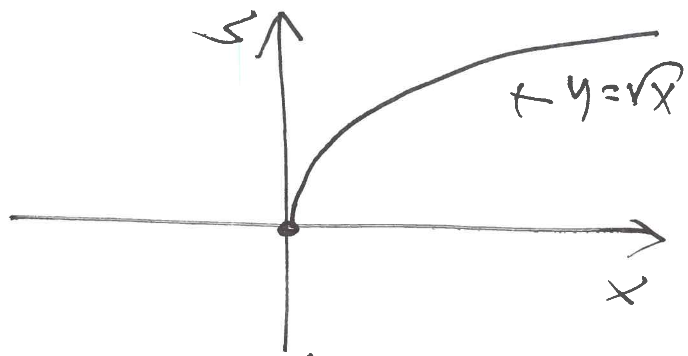


$$\frac{dy}{dx} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Example, $y = \sqrt{x}$

$\Updownarrow$

$y^2 = x$



At $x = 0$, tangent line is vertical, so no slope.

($y = x^{1/2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$ blows up at $x = 0$)

Fact: $\frac{d(x^n)}{dx} = nx^{n-1}$

If we can compute $f'(a)$ somehow, the line tangent to f at $(a, f(a))$ is $Y = f'(a)(X-a) + f(a)$

→ linear approx to f about a is $f(x) \simeq f(a) + (x-a)f'(a)$

(both are line of slope $f'(a)$ through point $(a, f(a))$)

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3. The tangent line

(6) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

(7) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

the point of tangency is $(1, 4 \cdot 1 + 2) = (1, 6)$

slope of line is 4

find which graphs pass through $(1, 6)$ with slope 4

4. Linear approximation

Definition. $f(a+h) \approx f(a) + f'(a)h$

(10) Estimate

(a) $\sqrt{1.2}$

Here let $f(x) = \sqrt{x}$ understand $f(1) = 1$; since $f'(x) = \frac{1}{2\sqrt{x}}$

so $f'(1) = \frac{1}{2}$

so linear approx is

$f(1+h) \approx f(1) + \frac{1}{2}h \approx 1 + \frac{1}{2}h$ so $f(1.2) \approx 1 + \frac{1}{2} \cdot 0.2 = 1.1$

(b) (Final, 2015) $\sqrt{8}$