4. Computing derivatives (27/9/2024)

Goals.

(1) Combining linear approximations
(2) Linearity of the derivative
(3) The product and quotient rules

Last Time. Linear approximation + Definition of the derivative

Typically, if f is defined near x, $f(x+h) - f(x) \sim ch$ (as $h \to 0$) for some constant c, equivalently $f(x+h) \approx f(x) + ch$

then we call c the **derivative** of f at x, write $f'(x) = c$, $\left.\frac{df}{dx}\right|_{x} = c$.

$\Rightarrow$ Line $y = f(a) + f'(a)h = f(a) + f'(a)(x-a)$ h

is tangent to f at a

$\Rightarrow$ $f(a+h) \approx f(a) + f'(a)h$, $f(x) \approx f(a) + f'(a)(x-a)$

is the linear approximation to f about a.

Aside: if $f(x+h) - f(x) \ll h$ then $f'(x) = 0$

Math 100A – WORKSHEET 4
COMPUTING DERIVATIVES

1. Review of the derivative

(1) Expand $f(x+h)$ to linear order in h for the following functions and read the derivative off:

(a) $f(x) = bx$

$f(x+h) = b(x+h) = bx + bh$

so $f'(x) = b$

Correct to 1st order

(b) $g(x) = ax^2$

$g(x+h) = a(x+h)^2 = ax^2 + 2axh + ah^2 \approx ax^2 + (2ax)h$

so $g'(x) = 2ax$.

(c) $H(x) = ax^2 + bx$.

Date: 25/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$H(x+h) = f(x+h) + g(x+h) \approx (bx+bh) + (ax^2 + (2ax)h)$

$\approx (ax^2+bx) + (2ax+b)h$ so $H'(x) = 2ax+b$.

(d) $i(x) = \frac{1}{b+x}$

$$\frac{1}{b+x+h} = \frac{1}{b+x} + \left(\frac{1}{b+x+h} - \frac{1}{b+x}\right) = \frac{1}{b+x} + \frac{-h}{(b+x)(b+x+h)}$$

$$\approx \frac{1}{b+x} - \frac{h}{(b+x)^2}$$

So $\quad i'(x) = -\frac{1}{(b+x)^2}$

(e) $j(x) = 4x^4 + 5x$ (hint: use the known linear approximation to $2x^2$)

by 1(b), $2(x+h)^2 \approx 2x^2 + (4x)\cdot h$

So $4(x+h)^4 = (2(x+h)^2)^2 \approx (2x^2+4x\,h)^2 \approx$

to approximate g^2, approximate g and square the answer.

$\approx 4x^4 + 16x^3h + 16x^2h^2$

$\approx 4x^4 + 16x^3h$ correct to 1st order in h

Also $5(x+h) \approx 5x + 5h$

So $j(x+h) \approx j(x) + (\underbrace{16x^3 + 5}_{j'(x) = 16x^3+5})h$

Summary + Diff Laws

① From linear approx $f(x+h) \approx f(x) + ch$ can read off $f'(x)$

② To approximate complicated expression, can approximate pieces, then put approximations together.

Suppose f, g are diff at x. $f(x+h) \approx f(x) + f'(x)h$
$g(x+h) \approx g(x) + g'(x)h$

Then $(f+g)(x+h) = f(x+h) + g(x+h) \approx (f(x) + f'(x)h) + (g(x) + g'(x)h)$
$\approx (f+g)(x) + (f'(x) + g'(x))h$

$\Rightarrow (f+g)'(x) = f'(x) + g'(x)$ $\left(\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} \right)$ linearity of the derivative (or: sum rule)

Similarly $(\alpha f)'(x) = \alpha \cdot f'(x)$

Then $(fg)(x+h) = f(x+h)g(x+h) \approx (f(x) + f'(x)h)(g(x) + g'(x)h)$
$\approx f(x)g(x) + f'(x)g(x)h + f(x)g'(x)h + f'(x)g'(x)h^2$

So $(fg)(x+h) \approx (fg)(x) + \left(f'(x)g(x) + f(x)g'(x)\right)h$

$$\Rightarrow (fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\frac{d(fg)}{dx} = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}.$$

product rule

Also: $\frac{f}{g}(x+h) \approx \frac{f}{g}(x) + \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\,h$

$$\Rightarrow \left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

quotient rule

(2) Differentiate

(a) $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

$$\frac{df}{dx} = 6\frac{d(x^{\pi})}{dx} + 2\frac{d(x^{e})}{dx} - \frac{d(x^{7/2})}{dx} = 6\pi x^{\pi-1} + 2e\,x^{e-1} - \frac{7}{2}x^{5/2}$$

linearity

power law rule

(b) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

$$\frac{d(x^2e^x)}{dx} = \frac{d(x^2)}{dx}\cdot e^x + x^2\frac{d(e^x)}{dx} = 2x\cdot e^x + x^2e^x = (2x+x^2)e^x$$

pdt rule

power law, exponential

$$\frac{d(x^ae^x)}{dx} = x^{a-1}(a+x)e^x.$$

(c) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$\frac{dh}{dx} \underset{\substack{\uparrow \\ \text{quot. rule}}}{=} \frac{(x^2+3)'\cdot(2x-1) - (x^2+3)(2x-1)'}{(2x-1)^2} = \frac{2x(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} =$$

$$= \frac{2x^2-2x-6}{(2x-1)^2}$$

(d) $\frac{x^2+A}{\sqrt{x}}$

① can use quot rule ② can write $\frac{x^2+A}{\sqrt{x}} = (x^2+A)\cdot x^{-\frac{1}{2}}$, use pdt rule

③ $\frac{x^2+A}{\sqrt{x}} = x^{3/2} + Ax^{-1/2}$ so $\frac{d\left(\frac{x^2+A}{\sqrt{x}}\right)}{dx} = \frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$.

takeaways: ① can have multiple approaches
② sometimes pays off to do algebra first.

(3) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

We have $f'(x) = \dfrac{(\sqrt{x}+A) - x\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+A)^2} = \dfrac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$

So $f'(4) = \dfrac{\frac{1}{2}\cdot 2 + A}{(2+A)^2}$ so $\dfrac{1+A}{(2+A)^2} = \dfrac{3}{16}$

so $16 + 16A = 3(A^2 + 4A + 4)$

so $\boxed{3A^2 - 4A - 4 = 0}$

Story

if we diff $f(x)$ plug in 4, set an expression involving A. Equating to $\frac{3}{16}$ would give an equation for A.

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.

$f(x) \approx f(1) + f'(1)(x-1) \approx 1 + 3(x-1)$

$g(x) \approx 2 + 4(x-1)$

so $(fg)(x) \approx (1+3(x-1))(2+4(x-1)) \approx 2 + 10(x-1) + 12(x-1)^2$

$\approx 2 + 10(x-1)$ ← prefer this form

(b) Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10$

$\left(\frac{f}{g}\right)'(1) = \dfrac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \dfrac{3\cdot 2 - 1\cdot 4}{2^2} = \dfrac{1}{2}.$

Lesson: diff rules not just about symbolic expressions are statements about each point.

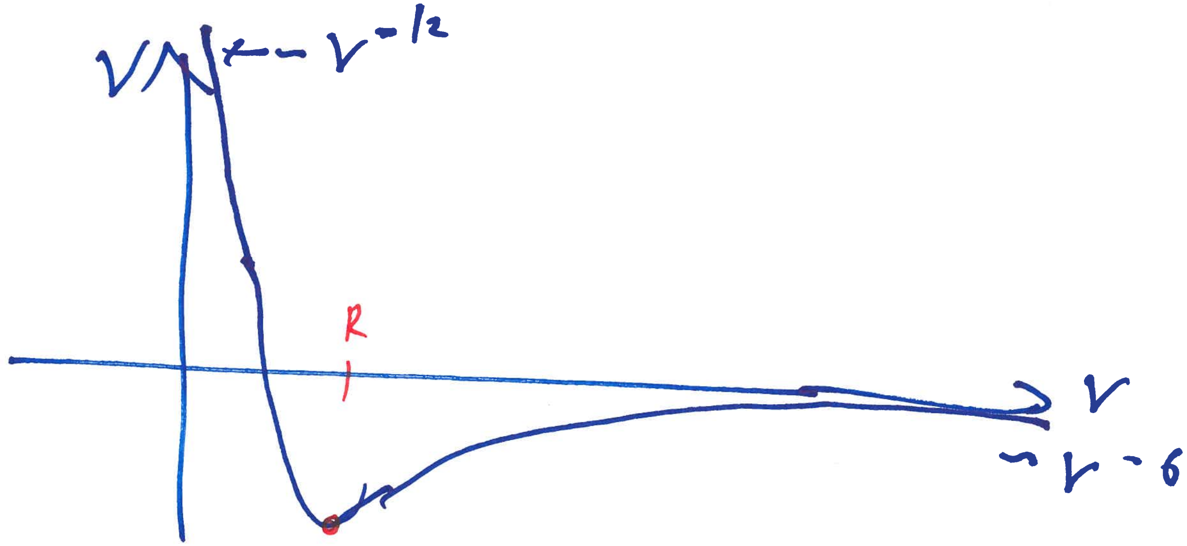
(6) The *Lennart–Jones potential* $V(r) = \epsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right)$ models the electrostatic potential energy of a diatomic molecule. Here $r > 0$ is the distance between the atoms and $\epsilon, R > 0$ are constants.

(a) What are the asymptotics of $V(r)$ as $r \to 0$ and as $r \to \infty$?

As $r \to 0$, $V(r) \sim \varepsilon R^{12} r^{-12}$

As $r \to \infty$ $V(r) \sim -2\varepsilon R^{6} \cdot r^{-6}$

(b) Sketch a plot of $V(r)$.



(c) Find the derivative $\frac{dV}{dr}(r) =$ $-12\varepsilon \frac{R^{12}}{r^{13}} + 12\varepsilon \frac{R^6}{r^7}$

$= 12\varepsilon \frac{R^6}{r^7}\left(1 - \frac{R^6}{r^6}\right)$

(d) Where is $V(r)$ increasing? decreasing? Find its minimum location and value.

$V'(r) > 0$ if $1 > \frac{R^6}{r^6}$ i.e. if $r > R$

$\Rightarrow$ increase if $r > R$

decrease if $r < R$