5. The Chain rule (2/10/2024)

Midterm next week (60 minutes)
do practise exam under exam condition

Goals.

(1) The Chain Rule: theory
(2) The Chain Rule: examples
(3) Logarithmic differentiation

Last Time. Diff rules, combining approximation:

(1) $$\frac{d(fg)}{dx} = \frac{df}{dx}\cdot g + f\frac{dg}{dx}$$ ← product rule

(2) $$\frac{d(f/g)}{dx} = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$$ ← quotient rule

(1) hold pointwise
(2) follow from multiplying/dividing linear approximations

Math 100A – WORKSHEET 5
THE CHAIN RULE

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.

(a) Expand $\sin(y+k)$ to first order in k. Write down the linear approximation to $\sin y$ about $y = a$.

$\sin(y+k) = \underbrace{\sin y}_{\text{"}f(y)\text{"}} + \underbrace{(\cos y)}_{\text{"}f'(y)\text{"}} k$, also $\sin(y) = \sin a + (\cos a)(y-a)$

(b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

by (a)

$F(x+h) = \sin(3(x+h)) = \sin(3x+3h) \approx \sin(3x) + (\cos 3x)\cdot(3h)$

$y = 3x$
$k = 3h$

$\approx F(x) + 3\cos(3x)\cdot h$ $\Rightarrow \frac{d}{dx}(\sin(3x)) = 3\cos(3x)$

In general, the composition of $f(y)$, $g(x)$ is the function $h = f \circ g$ defined by $h(x) = f(g(x))$

Chain rule: If $g'(x)$ exists, $f'(g(x))$ exists then $(f \circ g)'(x)$ exists and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or: if $y = g(x)$, $z = f(y)$, $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

Previous example: want $\frac{d(\sin(3x))}{dx} = \cos(3x) \cdot \frac{d(3x)}{dx} = \cos(3x) \cdot 3$

"chain rule"

the derivative of $\sin\theta$ is $\cos\theta$

In general, say $g(x+h) \approx g(x) + g'(x)\cdot h$

$$f(y+k) \approx f(y) + f'(y)k$$

Then $f(g(x+h)) \approx f(\underbrace{g(x)}_{y} + \underbrace{g'(x)h}_{k}) \approx f(g(x)) + f'(g(x))(g'(x)h)$

to 1st order in h $\approx f(g(x)) + (f'(g(x))g'(x))\cdot h$

(2) Write each function as a composition and differentiate

(a) e^{3x}

$f(g(x))$ where $f(y) = e^y$; $f'(y) = e^y$

$g(x) = 3x$; $g'(x) = 3$

So $\frac{d(e^{3x})}{dx} = e^{3x} \cdot 3$

(b) $\sqrt{2x+1}$

$f \circ g$ with $f(y) = y^{1/2}$

$g(x) = 2x+1$

so $\frac{d(\sqrt{2x+1})}{dx} = \frac{d(\sqrt{2x+1})}{d(2x+1)} \cdot \frac{d(2x+1)}{dx} =$

$= \frac{1}{2\sqrt{2x+1}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

(c) (Final, 2015) $\sin(x^2)$

Composition of $\sin\theta$

with $\theta = x^2$

So $\frac{d(\sin(x^2))}{dx} = \frac{d(\sin\theta)}{d\theta} \cdot \frac{d\theta}{dx}$

$= \cos(x^2) \cdot 2x.$

(d) $(7x + \cos x)^n$.

Let $z = y^n$ with $y = 7x + \cos x$

want answer in terms of x

Then $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = ny^{n-1}(7 - \sin x) = n(7x + \cos x)^{n-1}(7 - \sin x)$

chain rule

linearity

(3) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

By chain rule, $\frac{df}{dx} = g'(2\sin x) \cdot 2\cos x$

since no formula for g, answer involves g', whatever that is

So $f'\left(\frac{\pi}{4}\right) = g'\left(2\sin\frac{\pi}{4}\right) \cdot 2\cos\left(\frac{\pi}{4}\right) = g'\left(\frac{2}{\sqrt{2}}\right) \cdot 2 \cdot \frac{1}{\sqrt{2}} = g'(\sqrt{2}) \cdot \sqrt{2}$

$= \sqrt{2} \cdot \sqrt{2} = 2.$

(4) Differentiate

(a) a^x for fixed $a > 0$ (hint: $a = e^{\log a}$)

$$a^x = (e^{\log a})^x = e^{(\log a)\cdot x}$$

so $$\frac{d}{dx}(a^x) = e^{(\log a)\cdot x} \cdot \log a = a^x \cdot \log a$$

(b) $7x + \cos(x^n)$

$$\frac{d(7x + \cos(x^n))}{dx} \underset{\text{linearity}}{=} 7 + \frac{d}{dx}(\cos(x^n)) \stackrel{\text{chain rule}}{=} 7 - \sin(x^n)\cdot nx^{n-1}$$

(c) $e^{\sqrt{\cos x}}$

$$\frac{d}{dx}\left(e^{\sqrt{\cos x}}\right) = e^{\sqrt{\cos x}}\, \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)$$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

By chain rule, $f'(g(x)) \cdot g'(x) = 3x^2$

At $x=4$ this reads $f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$

so $g'(4) = \frac{48}{5}$

Logarithms

$\log y$ is defined as the x s.t. $e^x = y$.

($\log x$ is the y s.t. $e^y = x$)

Say $y = \log x$ want $\frac{dy}{dx}$.

can diff along curves

$\Updownarrow$

can massage identities $\rightarrow$ $x = e^y$.

so $1 = \frac{dx}{dx} = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \cdot \frac{dy}{dx} = e^y \cdot \frac{dy}{dx}$ $\leftarrow$ get $\frac{dy}{dx}$ from chain rule

so $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ so $\boxed{\frac{d(\log x)}{dx} = \frac{1}{x}}$

$\uparrow$

can solve for $\frac{dy}{dx}$

2. Logarithmic differentiation

(6) $\log\left(e^{10}\right) = 10$ $\qquad$ $\log(2^{100}) = 100\log 2$

(7) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} = \frac{1}{ax}\cdot a = \frac{1}{x}$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2+3t\right) = \frac{1}{t^2+3t}(2t+3)$

Or: $\log(ax) = \log a + \log x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^2\log(1+x^2) =$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$

$= 2x\log(1+x^2) + x^2\frac{1}{1+x^2}\cdot 2x$

$= 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$

(8) (Logarithmic differentiation) differentiate

log converts pdt to sum $y = (x^2+1) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

So $\log y = \log(x^2+1) + \log(\sin x) + (-\frac{1}{2})\log(x^3+3) + \cos x$

take $\frac{d}{dx}$: $\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$

solved for $\frac{dy}{dx}$

So $\frac{dy}{dx} = (x^2+1)\cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}\left(\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x\right)$ ← substitute y

(9) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) x^n

If $y = x^n$, $\log y = n \log x$

so diff both sides wrt x

get. $\frac{1}{y}\frac{dy}{dx} = n\frac{1}{x}$

So $\frac{dy}{dx} = \frac{n}{x} \cdot x^n = nx^{n-1}$

or: $\frac{d}{dx}(x^n) = x^n \cdot \frac{d}{dx}(\log(x^n))$

$= x^n \frac{d}{dx}(n \log x)$

$= x^n\left(\frac{n}{x}\right) = nx^{n-1}$.

(b) x^x : $\log(x^x) = x\log x$

so $(x^x)' = x^x(x\log x)' = x^x(\log x + 1)$

Or:

$x^x = e^{(\log x)\cdot x}$

so

$(x^x)' = e^{(\log x)x} \cdot (e^{(\log x)\cdot x})'$

$= x^x \cdot (\log x + 1)$

(c) $(\log x)^{\cos x}$

$\log(\log x^{\cos x}) = \cos x \cdot \log(\log x)$

so $\frac{d}{dx}((\log x)^{\cos x}) = (\log x)^{\cos x} \cdot \frac{d}{dx}(\cos x \cdot \log(\log x)) = (\log x)^{\cos x}\left(-\sin x \log\log x + \cos x \frac{1}{\log x}\frac{1}{x}\right)$

$= \left(\frac{\cos x}{x\log x} - \sin x \log\log x\right) \cdot (\log x)^{(\cos x)}$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x only.