7. Related Rates (16/10/2024)

Goals.

(1) Direct application of the chain rule
(2) Problem-solving

Last Time. Inverse Trig

(1) restrict domain: $\theta = \arcsin x$ is the angle s.t. $\sin\theta = x$, and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\theta = \arccos x$ is the angle s.t. $\cos\theta = x$, $\theta \in [0, \pi]$

$\theta = \arctan x$ " " " " $\tan\theta = x$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(2) simplify trig(arctrig x) via right triangle

One side x
One side 1
third side by Pythagoras

(3) $\Rightarrow$ diff $\arcsin x$ by diff along curve $\theta = \arcsin x \Leftrightarrow \sin\theta = x$

get: $1 = \frac{dx}{dx} = \frac{d(\sin\theta)}{dx} = \cos\theta \cdot \frac{d\theta}{dx} = \sqrt{1-x^2}\,\frac{d\theta}{dx}$

from our triangle

$$\Rightarrow \frac{d(\arcsin x)}{dx} = \frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Math 100A – WORKSHEET 7
APPLICATIONS OF THE CHAIN RULE

1. Related Rates 1: differentiation

(1) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $(1, \sqrt{3})$ we have $\frac{dy}{dt} = 1$. Find $\frac{dx}{dt}$.

Have $y(t)^2 = x(t)^3 + 2x(t)$ at all times

$\Rightarrow$ (diff both sides) wrt t: $2y \cdot \frac{dy}{dt} = 3x^2 \cdot \frac{dx}{dt} + 2\frac{dx}{dt}$

$\frac{d(y^2)}{dt} \to$ $\qquad$ $\frac{d(x^3+2x)}{dt}$

at given time: $2 \cdot \sqrt{3} \cdot 1 = (3 \cdot 1^2 + 2)\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{2\sqrt{3}}{5}$

(2) Air is pumped into a spherical balloon at the rate of $13\text{cm}^3/\text{s}$. How fast is the radius of the balloon changing when it is 15cm?

Call the radius of the balloon r, the volume of the air V. Then $V = \frac{4}{3}\pi r^3$.

So $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ $\Rightarrow 13 = 4\pi \cdot 15^2 \frac{dr}{dt}$ $\Rightarrow \frac{dr}{dt} = \frac{13}{900\pi} \cdot \frac{\text{cm}}{\text{sec}}$

Date: 16/10/2024, Worksheet by Lior Silberman.

Alternative for WS 1

Along curve $y^2 = x^3 + 2x$, have $2y\frac{dy}{dx} = 3x^2 + 2$

so $\frac{dy}{dx} = \frac{3x^2+2}{2y}$

so $\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{dx}{dy} = \frac{dy}{dt} \cdot \frac{2y}{3x^2+2}$

"Related rates"

Have a relation between variables (some identity)

If the variables are functions of another variable (say time)

Can diff the relation wrt the indep variable, to get a new relation involving the derivatives of the variables.

Sometimes we have to work the relation out.

(5) A closed rectangular box has sides of lengths 4, 5, 6cm. Suppose that the first and second sides are lengthening by $2\frac{\text{cm}}{\text{sec}}$ while the third side is shortening by $3\frac{\text{cm}}{\text{sec}}$.

(a) How fast is the volume changing?

Call sides x, y, z. Then volume is $V = xyz$

so $\frac{dV}{dt} = \frac{dx}{dt} yz + x \frac{dy}{dt} z + xy \frac{dz}{dt}$

so at given time, $\frac{dV}{dt} = 2 \cdot 5 \cdot 6 + 4 \cdot 2 \cdot 6 + 4 \cdot 5 \cdot (-3) = 48 \frac{\text{cm}^3}{\text{sec}}$

(b) How fast is the surface area changing?

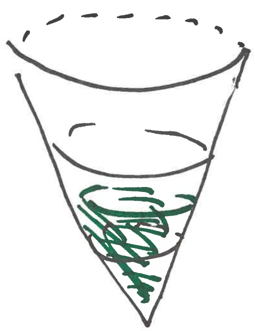
the area is $A = 2xy + 2xz + 2yz$

so $\frac{dA}{dt} = 2\left(\frac{dx}{dt} y + x \frac{dy}{dt} + \frac{dx}{dt} z + x \frac{dz}{dt} + \frac{dy}{dt} z + y \frac{dz}{dt}\right)$

(c) How fast is the main diagonal changing?

(6) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?



diagram

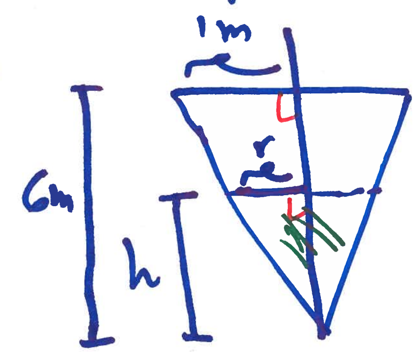
let V be the volume of the water,
h the height of the water
r the radius of top of water.

names

given $\frac{dV}{dt}$
asked dh/dt

then $V = \frac{1}{3}\pi r^2 h$

$\boxed{\text{Draw}}$ vertical cross-section.



by similiar triangles
$\frac{r}{h} = \frac{1}{6}$

combining relations

relations

so $r = \frac{1}{6}h$ so $V = \frac{\pi}{3 \cdot 6^2} h^3$ so $\frac{dV}{dt} = \frac{\pi}{36} h^2 \frac{dh}{dt}$ ← calculus

at given time $\frac{dV}{dt} = 5\,\frac{\text{m}^3}{\text{min}}$, $h = 5\text{m}$, so $\frac{dh}{dt} = \frac{180}{\pi \cdot 25} = \frac{36}{5\pi} \cdot \frac{\text{m}}{\text{min}}$

answer →

(b) The drain is unclogged and water begins to drain at the rate of $(5 + \frac{\pi}{4})\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

still have $\frac{dV}{dt} = \frac{\pi}{36} h^2 \frac{dh}{dt}$

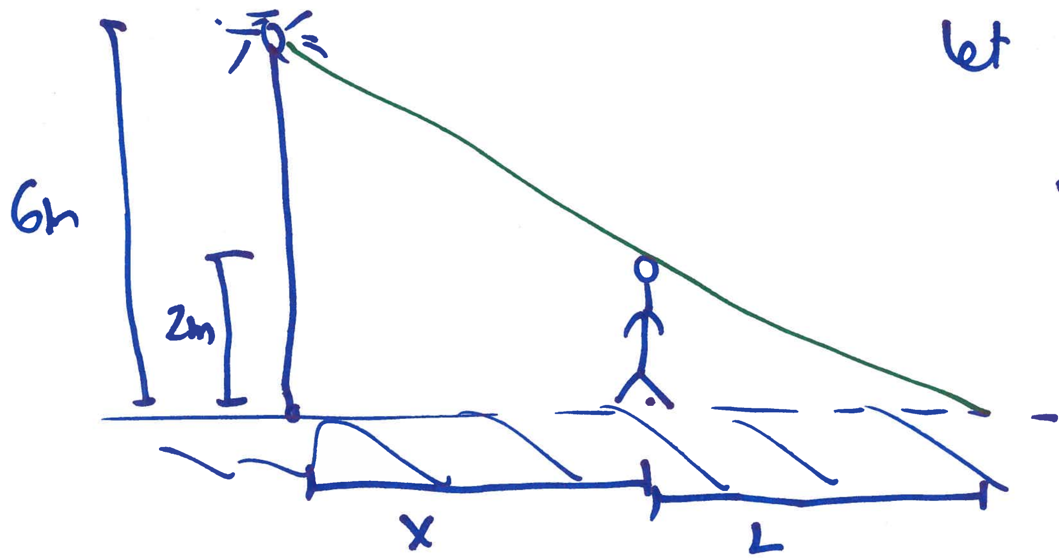
At given time $\frac{dV}{dt} = -\frac{\pi}{4} \frac{m^3}{min}$, $\frac{dh}{dt} = -1 \frac{m}{min}$

so $h^2 = \frac{36}{\pi} \cdot \left(-\frac{\pi}{4}\right)(-1) = 9\,m^2$ so $h = 3m$ at given time.

right order of ~~mag~~ magnitude

(c) Repeat the problem with tank upside-down (vertex on top).

(7) (Final, 2019) A 2m tall woman is running at night, moving away from a 6m-tall lamp-post. Her velocity t seconds after leaving the lamp-post is given (in metres per second) by $v(t) = 4 - \sin(2\pi t)$. How quickly is the length of her shadow changing after 3 seconds?



let $L(t)$ be the length of the shadow

$x(t)$ her distance from the lamp

Given $\frac{dx}{dt} = v(t)$, want $\frac{dL}{dt}$:

By similar triangles, $\frac{L}{2} = \frac{L+x}{6}$ so $6L = 2L + 2x$

so $2L = x$, $L = \frac{1}{2}x$.

so $\frac{dL}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}v(t)$

at $t=3$, $\frac{dL}{dt} = \frac{1}{2}(4 - \sin 6\pi) = 2 \frac{m}{sec}$