8. Curve sketching (23/10/2024)

Midterm grades
(1) see Tests page on Canvas
(2) Come to office hours

Goals.
(1) Convexity
(2) Curve sketching

Last Time. Related Rates = chain rule + problem-solving

~~When~~ Convexity

On interval

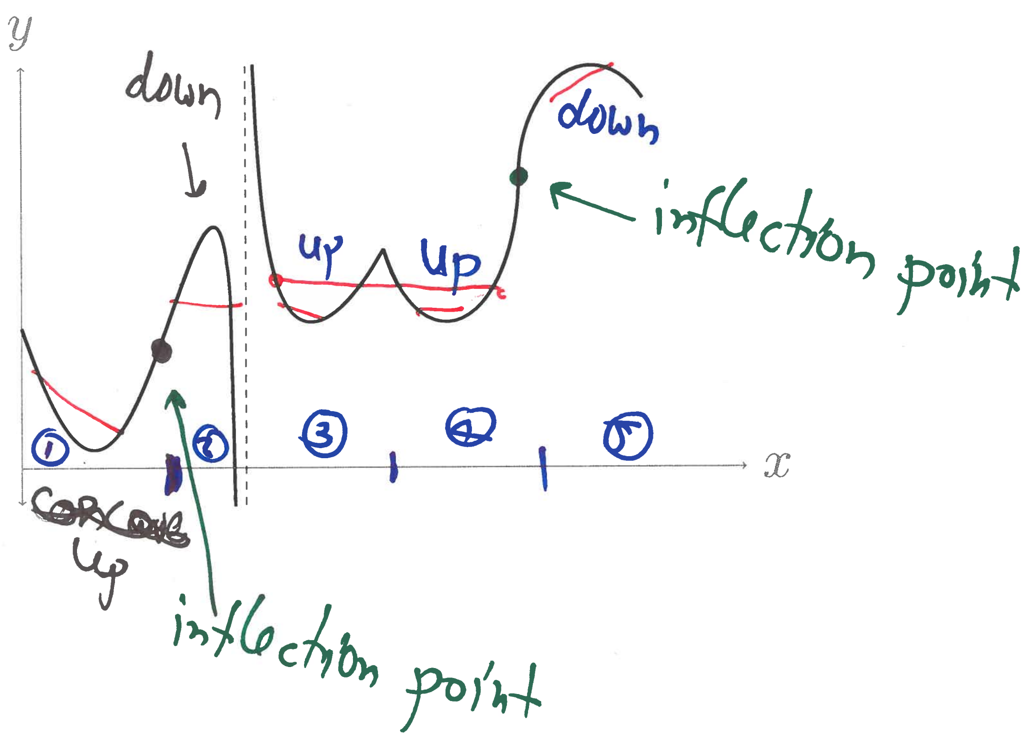
~~When~~ Consider graph of $y=f(x)$

① When $f''(x)>0$, the tangent line is below graph on interval, secant lines are above ← say f is convex or concave up

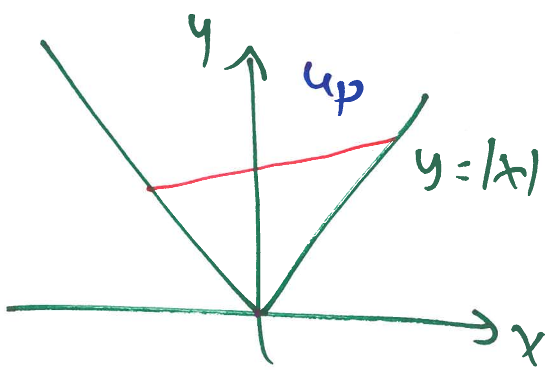
② When $f''(x)<0$, the tangent line is above graph, on interval, secant lines are below. ← say f is concave or concave down

③ If concavity changes at $(x_0, f(x_0))$ and f is cts there call this an inflection point

Example:


y
down
up
up
down
inflection point
1
2
3
4
5
x
~~concave~~
up
inflection point

But


y
up
y = |x|
x

(2) For each curve find its domain; where is it concave up or down? Where are the inflection points.

(a) $y = x \log x - \frac{1}{2}x^2$.

defined on $\{x > 0\} = (0, \infty)$

$\frac{dy}{dx} = \log x + 1 - x$, $\frac{d^2y}{dx^2} = \frac{1}{x} - 1$, want: when is $y'' > 0$, $y'' < 0$.

$\frac{1}{x} - 1 = \frac{1-x}{x} > 0$ if $1 - x > 0$ if $x < 1$

meeting point of interval

so concave up on $(0,1)$, down on $(1,\infty)$, inflection point $(1, -\frac{1}{2})$

(b) $y = \sqrt[3]{x}$.

defined on $(-\infty, \infty)$

$y' = \frac{1}{3}x^{-2/3}$, $y'' = -\frac{2}{9}x^{-5/3}$ positive if $x < 0$, negative if $x > 0$

so concave up on $(-\infty, 0]$, down on $[0, \infty)$

inflection point at $(0,0)$

IF f'' defined at inflection pt, then $f''(x_0) = 0$

Summary

$f''(x) > 0 \Rightarrow$ concave up

$f''(x) < 0 \Rightarrow$ concave down

- Inflection point can only occur where $f''(x_0) = 0$ OR $f''(x_0)$ DNE.

(but doesn't have to)

Curve sketching

Examples $y = x^4 - 4x^3 = x^3(x-4)$

0. $y = 0$ if $x = 0$, $x = 4$, on $(-\infty, 0)$ $y > 0$

on $(0, 4)$ $y < 0$

on $(4, \infty)$ $y > 0$

as $x \to \pm\infty$, $y \sim x^4$, $y(0) = 0$

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1. $\frac{dy}{dx} = 4x^3 - 4 \cdot 3x^2 = 4x^2(x-3)$

$y' = 0$ at $x = 0, 3$ critical points $(0,0)$, $(3, -27)$

$y' > 0$ if $x > 3$, $y' < 0$ on $(-\infty, 0)$, $(0, 3)$

1 Curve Sketching notes

1.1 Tools. Let f be differentiable as needed on (a,b).

Fact (First derivative). (1) *If $f'(x) > 0$ for all $x \in (a,b)$ then f is strictly* increasing *there.*
(2) *If $f'(x) < 0$ for all $x \in (a,b)$ then f is strictly* decreasing *there.*

Every change involves either a *critical point* (f' vanishes) or a *singularity* (f' undefined).

Fact (Second derivative). (1) *If $f''(x) > 0$ for all $x \in (a,b)$ then f is* concave up *there.*
(2) *If $f''(x) < 0$ for all $x \in (a,b)$ then f is* concave down *there.*

Definition. A change in concavity is called an *inflection point.*

Theorem. *(Tests for minima and maxima) Let $x_0 \in (a,b)$ be a critical or singular number for f, and suppose f is continuous at x_0, differentiable near it.*

(1) *Either of the following is sufficient to show that f has a local* minimum *at x_0:*
  (a) $f''(x_0) > 0$ *or;*
  (b) *$f'(x)$ is negative to the left of x_0, positive to its right.*
(2) *Either of the following shows that f has a local* maximum *at x_0:*
  (a) $f''(x_0) < 0$ *or;*
  (b) *$f'(x)$ is positive to the left of x_0, negative to its right.*

1.2 Curve sketching protocol. Given a function f.

0th derivative stuff:
  (a) The domain and the domain of continuity.
  (b) Domains where $f > 0$, $f < 0$.
  (c) Anchor points: x- and y-intercepts.
  (d) vertical asymptotes.
  (e) Asymptotics at $\pm\infty$ (if in the domain)

1st derivative stuff:
  (a) Evaluate $f'(x)$ **[high stakes: error here loses a lot of points down the line]**
  Using this, determine:
  (b) Domains where $f' > 0$, $f' < 0$
  (c) Critical and singular points.

2nd derivative stuff:
  (a) Domains where $f'' > 0$, $f'' < 0$
  (b) Points where $f''(x) = 0$, inflection points.

decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

so $(3, -27)$ is a local minimum.

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(2) $\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2)$ vanishes at $x = 0, 2$

$y'' > 0$ on $(-\infty, 0)$ [there $x < 0$, $x - 2 < 0$]

$y'' < 0$ on $(0, 2)$ [$x > 0$, $x - 2 < 0$]

$y'' > 0$ on $(2, \infty)$ [$x > 0$, $x - 2 > 0$]

so concave up on $(-\infty, 0)$, $(2, \infty)$, concave down on $(0, 2)$

so inflection points $(0, 0)$, $(2, -16)$

Math 100A – WORKSHEET 8
CURVE SKETCHING

1. Convexity and Concavity

(1) Consider the curve $y = x^3 - x$.

(a) Find the line tangent to the curve at $x = 1$.

$y'(1) = [3x^2 - 1]_{x=1} = 2$ so line is $y = 2(x-1)$

(b) Near $x = 1$, is the line above or below the curve? Hint: how does the slope of the curve behave to the right and left of the point?

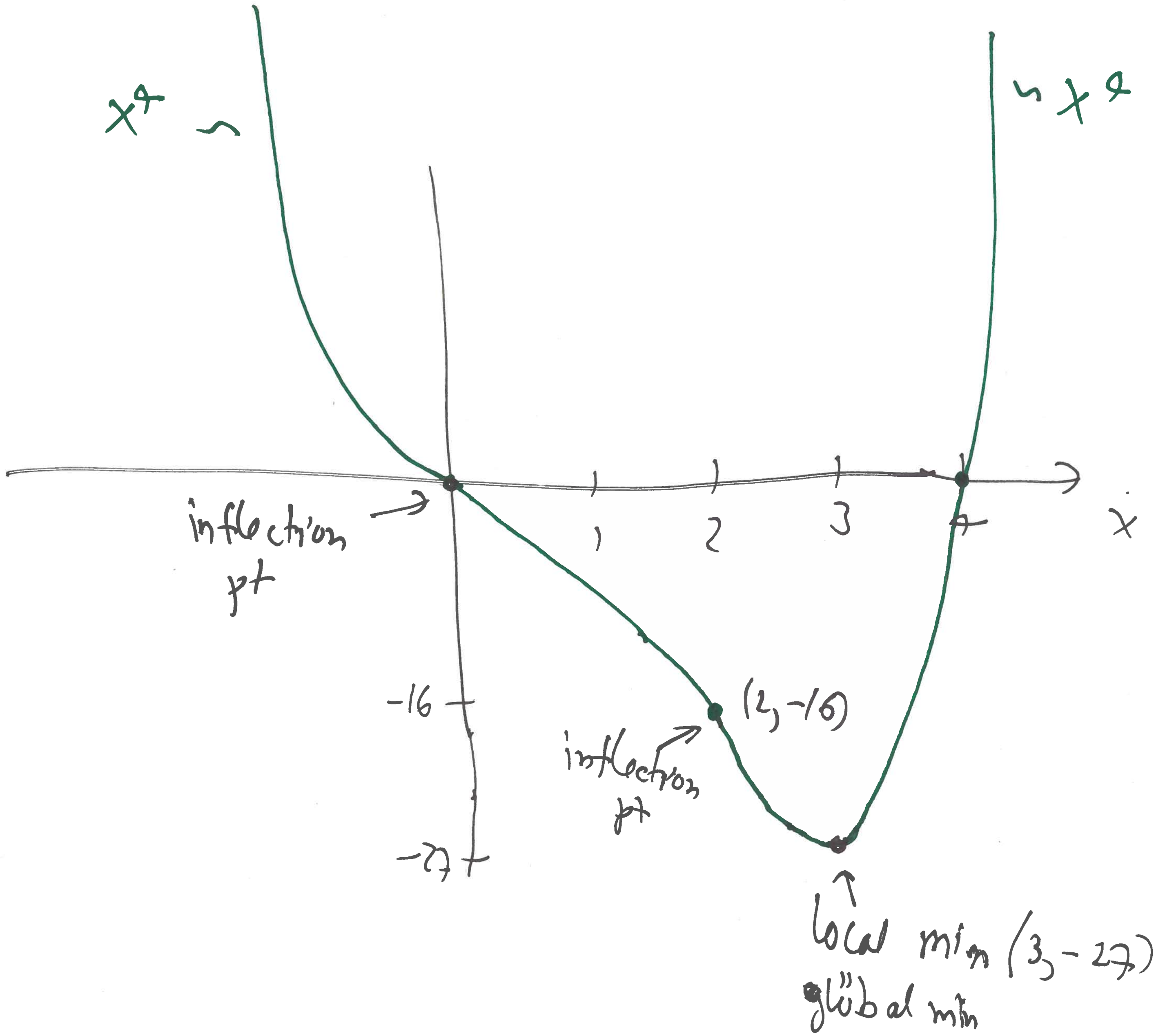
To right of 1, $y' > 2$ so curve growing faster, to left of 1, $y' < 2$ so line falling faster. $\Rightarrow$ line is below curve.

$y'' > 0 \Rightarrow$ tangent line is below curve

Date: 23/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

calculate: $x^3 - x - 2(x-1) = (x-1)(x^2+x-2) = (x-1)\left((x-1) + (x-1)(x-1+2)\right)$

$= 2(x-1)^2 + (x-1)^3 \sim 2(x-1)^2 > 0$ as $x \to 1$



x^4 ~
~ x^4
inflection pt
1
2
3
4
x
-16
-27
$(2, -16)$
inflection pt
local min $(3, -27)$
global min

2. CURVE SKETCHING

(3) Let $f(x) = \frac{x^2}{x^2+1}$ for which $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotics there?

Since $x^2+1>0$ always, f is defined on $(-\infty,\infty)$. $f>0$ except at $(0,0)$

As $x\to\infty$, $f(x) \sim \frac{x^2}{x^2} \to 1$ same as $x\to-\infty$ No vertical asymptotes.

(b) What are the intervals of increase/decrease? The local and global extrema?

Since $(x^2+1)^2>0$, $f'>0$ where $x>0$ so

f ~~[illegible]~~ decreasing on $(-\infty,0]$, increasing on $[0,\infty)$

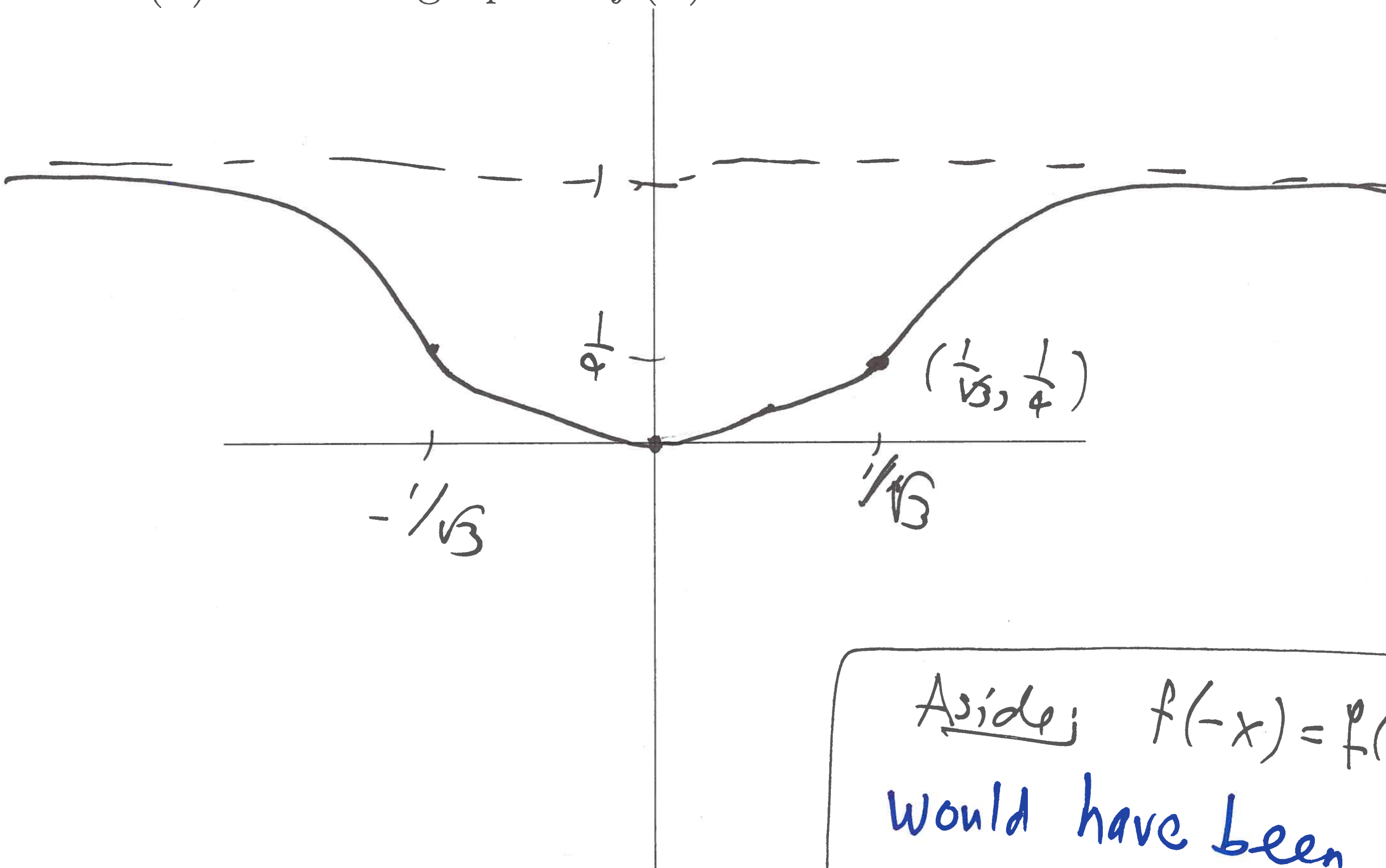
so $(0,0)$ is the global minimum (also a local min) inflection pts $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$

(c) What are the intervals of concavity? Any inflection points?

$(x^2+1)^3>0$ so $f''>0$ when $1-3x^3>0$, when $3x^2<1$, when $|x|<\frac{1}{\sqrt{3}}$

so f is concave down on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \infty\right)$, concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(d) Sketch a graph of $f(x)$.



Aside: $f(-x)=f(x)$, so f is <u>even</u>

would have been enough to compute on $[0,\infty)$

true: as $x \to \infty$, $(x-\mu)^2 \sim x^2$
but $e^{-\frac{(x-\mu)^2}{2\sigma^2}} \neq e^{-x^2/2\sigma^2}$: ratio is $e^{\frac{2x\mu}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}}$

(4) Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotics there?

domain $(-\infty, \infty)$, $f > 0$ for all x since $e^u > 0$ for all u.

no simpler asymptotics but $\lim_{x \to \pm\infty} f(x) = 0$ (horizontal asymptote!)

• $-\frac{(x-\mu)^2}{2\sigma^2}$ is a large negative number

no vertical asymptotes

(b) What are the intervals of increase/decrease? The local and global extrema?

$$f'(x) = -\frac{1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)$$

Since $e^u > 0$ for all u, $f'(x) > 0$ on $(-\infty, \mu)$ increase
$f'(x) < 0$ on (μ, ∞) decrease

critical pt at $\left(\mu, \frac{1}{\sqrt{2\pi\sigma^2}}\right)$, which is the global maximum

$$f''(x) = \frac{1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(\left(\frac{x-\mu}{\sigma}\right)^2 - 1 \right)$$

(c) What are the intervals of concavity? Any inflection points?

$f'' > 0$ if $\frac{(x-\mu)^2}{\sigma^2} - 1 > 0 \iff |x-\mu| > \sigma$ so concave up $(-\infty, \mu-\sigma)$ and $(\mu+\sigma, \infty)$

down on $(\mu-\sigma, \mu+\sigma)$

inflection pts at $x = \mu \pm \sigma$

$\left(\mu \pm \sigma, \frac{1}{\sqrt{2\pi e \sigma^2}}\right)$

(d) Sketch a graph of $f(x)$.