9. Optimization (1/11/2024)

Goals.

(1) Review: calculus and the shape of the graph
(2) Optimization of functions
(3) Problem solving: optimization problems

Last Time. Curve Sketching

Idea: use asymptotics, f', f'' to get a reasonable description of graph of $y=f(x)$.

l'Hôpital's rule: When studying limit $\lim_{x\to a}\frac{f(x)}{g(x)}$

If
1. $f,g \xrightarrow[x\to a]{} \pm\infty$ or $f,g \xrightarrow[x\to a]{} 0$
2. f', g' exist near a
3. $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$ exists

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = L$

(also true if a is $\pm\infty$
L "is" $\pm\infty$)

Recall

$f'(x) > 0 \Rightarrow f$ increasing near x

$f'(x) < 0 \Rightarrow f$ decreasing near x

$\Rightarrow f$ can have a local max/min at x only if

$f'(x) = 0$ or $f'(x)$ DNE

$(x, f(x))$ critical point $\quad$ $(x, f(x))$ is a singular point

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Local extremum: $f(x)$ larger/smaller than values near the point

Can know that a point is a local min/max if ① f increasing/decreasing on both sides (from signs of f')

② $f''(x) > 0$ (local min)

$f''(x) < 0$ (local max)

Caveat: at endpoints f can have local max/min without restriction on f'.

Global max/min

Goal of optimization: find largest/smallest value of f in some range of indep variable.

Necessarily occurs at some local extremum (if it exists)

Promise: If f is cts on closed interval $[a,b]$ then it has a global max & min

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⇒ Closed interval method

If f cts on $[a,b]$: ① find all critical, singular, end points

② evaluate f at those points

③ choose largest/smallest value

Math 100A – WORKSHEET 9
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

$f'(x) = 4x^3 - 8x = 4x(x^2-2)$ vanishes at $x = 0, \pm\sqrt{2}$

$f(\pm 5) = 625 - 100 + 4 = 529$, $f(\pm\sqrt{2}) = 0$, $f(0) = 4$

max $= 529$, min $= 0$

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

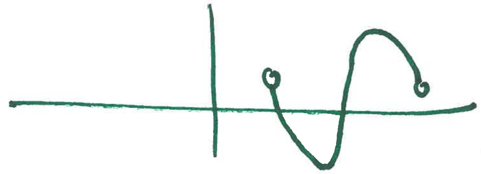
Here f' only vanishes at 0 ($\pm\sqrt{2}$ outside $[-1,1]$)

and $f(\pm 1) = 1$, $f(0) = 4$

max $= 4$, min $= 1$

Point: make sure to only work <u>in</u> the domain

Date: 30/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.



← Can have max & min on open interval

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

still $\max = 4$

no min (as $x \to \pm 1$, $f(x) \to 1$ but never gets there)

<u>Point:</u> on open domain might not have global max/min

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

Still have critical pts at $0, \pm\sqrt{2}$.

Now as $x \to \pm\infty$ $f(x) \sim x^4 \to \infty$ $\Rightarrow$ no max

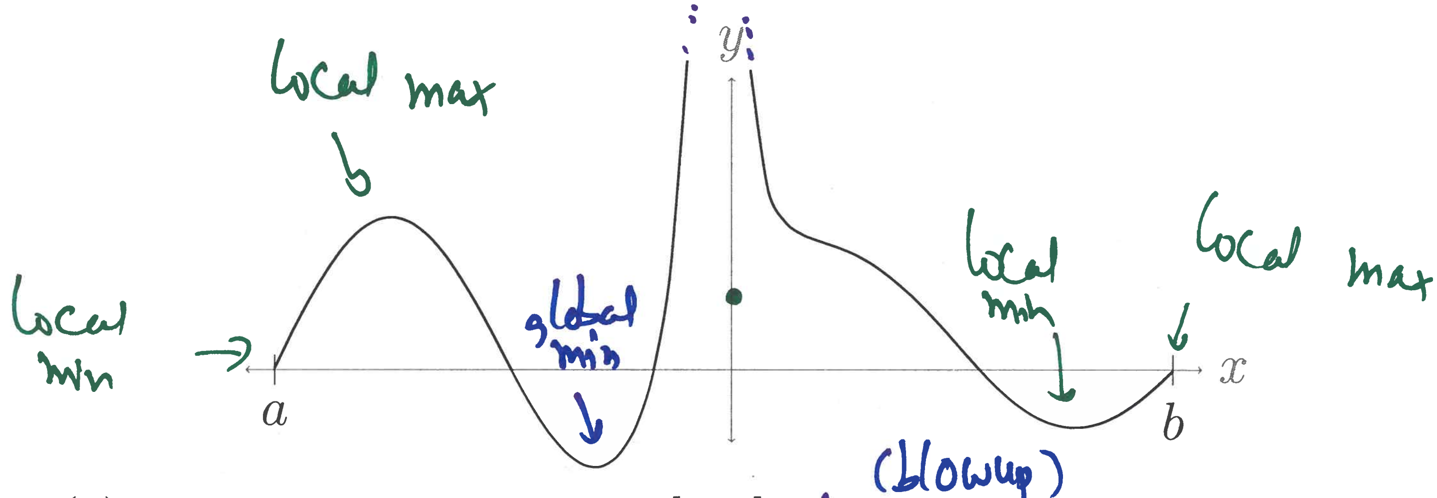
but also (f is cts) must have a global min (0 at $\pm\sqrt{2}$)

<u>Point:</u> ① Can use asymptotics to handle open endpoints

② If $f \to \infty$ and endpoint, no max, but if $\to \infty$ at both endpoints, ~~can~~ will have min on some closed interval.

reasoning why f has min →

Examples



(1) Is there a *global* max on $[a, b]$? no Is there a global min? yes

(2) How many *local* maxima are there? 2 local minima? 3

Which point on the line $3x + y = 6$ is closest to the point $(7, 5)$?

distance from (x,y) to $(7,5)$ is $\left((x-7)^2+(y-5)^2\right)^{1/2}$

Find point minimizing $f(x) = (x-7)^2 + (y-5)^2$ ← easier to work with $(\text{dist})^2$

$= (x-7)^2 + (6-3x-5)^2$ ← enforced relation (x,y) on the line

$= 10x^2 - 20x + 50$

$= 10(x^2 - 2x + 5)$

as $x \to \pm\infty$ $f(x) \sim 10x^2 \to \infty$

f cts so min at critical pt

$f'(x) = 10(2x-2)$ so only crit pt at $x=1$, so closest point is $(1,3)$ ← answer question

(4) A fish swimming at speed v relative to the water faces a drag force of the form av^2 and thus has to output a power of av^3. If the fish is swimming against a current of speed $u > 0$ (thus with speed $v > u$), it will cover a distance L at time $\frac{L}{v-u}$. The total energy cost is then $E = av^3 \frac{L}{v-u}$. At what speed v should the fish swim to minimize this cost?

Need to minimize $E(v) = aL\frac{v^3}{v-u}$ on (u, ∞), this is ctn, diff everywhere

As $v \to u$, $E(v) \sim aLu^3 \frac{1}{v-u} \to \infty$, As $v \to \infty$, $E(v) \sim aLv^2 \to \infty$

so min is in the interior, at a critical pt.

$E'(v) = aL\frac{3v^2(v-u) - v^3}{(v-u)^2} = aL\frac{v^2(2v-3u)}{(v-u)^2}$ vanishes at $\boxed{v = \frac{3}{2}u}$ ($v=0$ not in domain)

(5) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

(a) What is the range of r values that makes sense?

tips: can have parameters (a, L, u)
indep var doesn't have to be x.

Optimization / Related Rates notes

(0) Read problem: understand the idea, draw a picture if possible.
(1) Assign names:
- Choose axes, quantities of interest.
- Give a *name* to each quantity of interest.

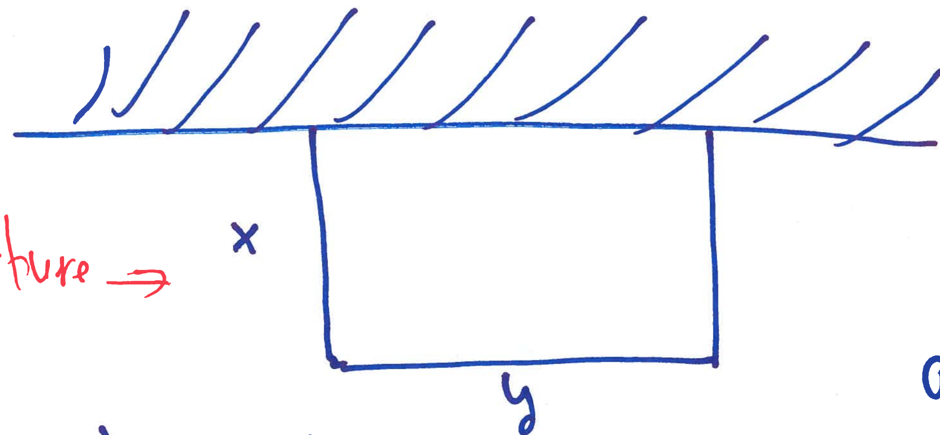
(2) Function/relations: express quantity to be optimized as a function of the dependent variable.
- Sometimes the quantity depends on several variables, and we need to enforce *relations* between them to end up with one independent variable.

(3) Calculus: find the (relevant) domain of the objective function and the minima and maxima on the domain.
- (Related rates: use the chain rule when differentiating).

(4) Interpretation: solve the problem using the calculus result.
- Make *sanity checks* (area can't be negative, for example).

(6) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?

picture →



~~Then~~ Let x, y be the sides of the rectangle. Then the area is $A = xy$.

name

relations

we have the constraint: $100 = 2x + y$, so $y = 100 - 2x$,

objective function → $A = x(100-2x) = 2x(50-x)$

defined on $[0, 50]$: need $x \geq 0$, $y \geq 0$

domain

$A(0) = 0$, $A(50) = 0$

$A'(x) = 100 - 4x$, only crit pt at $x = 25$, $A(25) = 1250\, m^2$

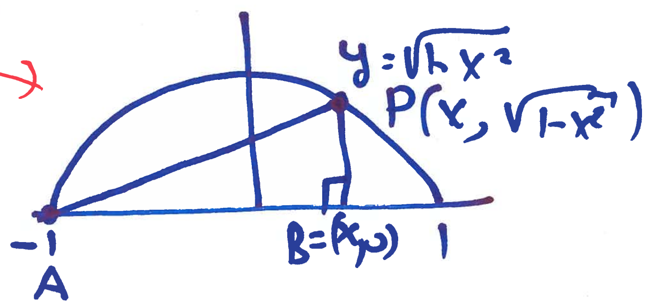
so largest possible area is $1{,}250\, m^2$.

Aside: $100x - 2x^2 = -2(x^2 - 50x) = -2(x^2 - 50x + 625) + 2 \cdot 25^2 = 1250 - 2(x-25)^2 \le 1250.$

Advice: Admitting degenerate solutions:
the 0×100, 50×0 "rectangles"
means we get to optimize on closed interval

(8) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?

picture



objective fcn

Here area is $f(x) = \frac{1}{2}(1+x)\sqrt{1-x^2}$ defined on $[-1,1]$ ← domain

$f(-1) = f(1) = 0$, f cts, $f \geq 0$, so max is in interior. ← closed interval method

calculus

$$f'(x) = \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}(1+x)\frac{x}{\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}}\left(1-x^2-x-x^2\right)$$

$$= \frac{1}{2\sqrt{1-x^2}}\left(1-x-2x^2\right)$$ vanishes at $x = \frac{1 \pm \sqrt{1+8}}{-4} = \frac{1 \pm 3}{-4} = -1, +\frac{1}{2}$

so crit pt at $x = +\frac{1}{2}$, where

$$\boxed{f\left(\tfrac{1}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\sqrt{1-\left(\tfrac{1}{2}\right)^2} = \frac{3\sqrt{3}}{8}}$$