10. Taylor expansion (4/11/2024)

Goals.

(1) Review: Linear approximation
(2) Higher order approximation
(3) Manipulating expansions

Last Time. Optimization

"Closed interval method": let f defined on $[a,b]$; (1) if f cts, it has a max & a min

(2) these occur at one of the critical, singular, or end points
$f'(x_0)=0$, $f'(x)=DNE$ a,b.

Use asymptotics to handle "end" "points" of open intervals (including ∞)

constant approx: $f(x) \approx f(a)$ ← continuity

Linear approx: "near" a, $f(x) \approx f(a) + f'(a)(x-a)$

To create this, identify the "limit" a, set small parameter $x-a$, use derivative to find coefficient $f'(a)$.

Different choice:

Let $f(x) = \log x$ $\quad f(1) = 0 \Rightarrow \log x \approx 0 + 1\cdot(x-1)$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

Math 100A – WORKSHEET 10
TAYLOR EXPANSION

1. Taylor expansion

(1) (Review) Use linear approximations to estimate:

(a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

Let's work near 1: let $f(u) = \log(1+u)$, Then $f(0) = \log 1 = 0$

$f'(u) = \frac{1}{1+u}$ $\quad f'(0) = \frac{1}{1+0} = 1$

so $f(u) \approx f(0) + f'(0)u \approx u$, so $\log\left(\frac{4}{3}\right) = \log\left(1+\frac{1}{3}\right) \approx 1/3$, $\log\left(\frac{2}{3}\right) = \log\left(1-\frac{1}{3}\right) \approx -\frac{1}{3}$

(b) $\sin 0.1$ and $\cos 0.1$.

If $s(\theta) = \sin\theta$, $s'(\theta) = \cos\theta$, $s(0) = 0$, $s'(0) = 1$, $\sin\theta \approx \theta$ to 1^{st} order

$\cos 0 = 1$, $-\sin 0 = 0$, so $\cos\theta \approx 1 + 0\cdot\theta = 1$ to 1^{st} order.

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

1a) $f^{(1)}(x)=e^x$, $f^{(2)}(x)=e^x$, $f^{(3)}(x)=e^x$, $\ldots$ $f^{(k)}(x)=e^x$ ← notation for k th derivative

So $f(0)=1$, $f^{(1)}(0)=1$, $f^{(k)}(0)=e^0=1$.

(b) $T_0(x)\equiv 1$ has $T_0(0)=f(0)$

(c) $T_1(x)=1+x$ ← linear correction

const approx

(d) guess $T_2(x)=1+x+cx^2$

Then $T_2(0)=1$,

$T_2'(x)=1+2cx$, $T_2'(0)=1$

$T_2''(x)=2c$ so to make $T_2''(0)=$

choose $c=\frac{1}{2}$.

get $T_2(x)=1+x+\frac{1}{2}x^2$

quadratic correctio

(e) Try $T_3(x)=1+x+\frac{1}{2}x^2+dx^3$

dx^3, $(dx^3)'$, $(dx^3)''$ all vanish at $x=0$

$3dx^2$, $2\cdot 3\cdot xd$ $\quad (dx^3)^{(3)}=1\cdot 2\cdot 3\cdot d$

so to get 1, need $d=\frac{1}{1\cdot 2\cdot 3}$, $T_3(x)=1+1\cdot x+\frac{1}{1\cdot 2}x^2+\boxed{\frac{1}{1\cdot 2\cdot 3}x^3}$

Recap

As $x \to 0$, (0) $e^x \sim 1$ correct to 0^{th} order

(1) $e^x - 1 \sim x \Rightarrow e^x \approx 1 + x$ correct to 1^{st} order

(2) $e^x - (1+x) \sim \frac{1}{2}x^2$, $\Rightarrow e^x \approx 1 + x + \frac{1}{2}x^2$ correct to 2^{nd} order

(3) $e^x - (1+x+\frac{1}{2}x^2) \sim \frac{1}{6}x^3 \Rightarrow e^x \approx 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$ correct to 3^{rd} order

plug in $x=1$:

$$\left.\begin{array}{llll} \text{Const approx} & \text{gives} & 1 & \\ \text{linear} & " & 1+1 & = 2 \\ \text{quadratic} & \text{gives} & 1+1+\frac{1}{2} & = 2\frac{1}{2} \\ \text{cubic} & \text{gives} & 1+1+\frac{1}{2}+\frac{1}{6} & = 2\frac{2}{3} \end{array}\right\} \text{Successive approximation to } f(1) = e^1 = e$$

Observation

Coeff of 1 was $f(0)$

Coeff of x was $\frac{1}{1}f'(0)$

Coeff of x^2 was $\frac{1}{1\cdot 2}f^{(2)}(0)$

Coeff of x^3 was $\frac{1}{1\cdot 2\cdot 3}f^{(3)}(0)$

$\vdots$

Coeff of x^k would be $\frac{1}{1\cdot 2\cdot 3\cdots k}f^{(k)}(0)$

Defs

$k! = 1\cdot 2\cdot 3\cdot \cdots \cdot k$

Called "k factorial"

$0! = 1$	$4! = 24$
$1! = 1$	$5! = 120$
$2! = 2$	$6! = 720$
$3! = 6$	$7! = 5040$

$\vdots$

Same would work if we worked as $x \to a$, with $a \neq 0$:

$$f(x) \approx f(a) + \frac{1}{1!}f^{(1)}(a)\cdot(x-a) + \frac{1}{2!}f^{(2)}(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \dots +$$

$$f(a+h) = f(a) + \frac{1}{1!}f^{(1)}(a)h + \frac{1}{2!}f^{(2)}(a)\cdot h^2 + \frac{1}{3!}f^{(3)}(a)h^3 + \dots$$

if we stop at k'th term, approx correct to k'th order, error decays like h^{k+1}. (more in next small class)

common errors

① "nonlinear line": putting in formula for $f^{(k)}(x)$ ← variable point
instead ~~[illegible]~~ of value $f^{(k)}(a)$ ← center of expansion

e.g. "linear approx to $\sin\theta$ about $\theta = 0$"

is NOT ~~is~~ $(\cos\theta)\cdot\theta$ but $(\cos 0)\cdot\theta$

derivative, θ — derivative at $\theta=0$

sanity check: is what we wrote a polynomial of the right degree?

② only writing kth order term

"cubic approx to e^x about $x=0$" is $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

NOT $\frac{1}{6}x^3$

approximation

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) ⋆ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

let $f(x) = (1-x)^{-1}$ $\qquad f(0) = 1$

$f^{(1)}(x) = (1-x)^{-2}$ $\qquad f^{(1)}(0) = 1$

$f^{(2)}(x) = 1\cdot 2(1-x)^{-3}$ $\qquad f^{(2)}(0) = 1\cdot 2$

$f^{(3)}(x) = 1\cdot 2\cdot 3(1-x)^{-4}$ $\qquad f^{(3)}(0) = 1\cdot 2\cdot 3$

$f^{(4)}(x) = 1\cdot 2\cdot 3\cdot 4(1-x)^{-5}$ $\qquad f^{(4)}(0) = 1\cdot 2\cdot 3\cdot 4$

So $T_4(x) = 1 + \frac{1}{1}x + \frac{1\cdot 2}{1\cdot 2}x^2 + \frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3}x^3 + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4}x^4$

$2!$ $\quad 3!$ $\quad 4!$

$= 1 + x + x^2 + x^3 + x^4$

Memorize

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin\phi = \phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \frac{1}{7!}\phi^7 + \cdots$$

$$\cos\phi = 1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\log(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \frac{1}{5}u^5 - \cdots$$

$$\Updownarrow$$

$$\log x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

(5) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

To 3rd order, $\cos \phi \approx 1 - \frac{1}{2}\phi^2$

$\cos 0.1 \approx 1 - \frac{1}{200} \approx \frac{199}{200}$.

(6) (Final, 2015) Let $T_3(x) = 24+6(x-3)+12(x-3)^2+4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

$12 = \dfrac{f''(3)}{2!}$ so $\boxed{f''(3) = 24}$ (or calculate $T_3''(3)$)

(7) In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the "rest mass" of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

Want to approx $E(v)$ for small v. Notice: If $f(x) = mc^2(1-x)^{-\frac{1}{2}}$

then $E(v) = f(\frac{v^2}{c^2})$, $f'(x) = \frac{1}{2}mc^2(1-x)^{-3/2}$, $f''(x) = \frac{3}{4}mc^2(1-x)^{-5/2}$

$f(0) = mc^2$, $f'(0) = \frac{1}{2}mc^2$, $f''(0) = \frac{3}{4}mc^2$

So $f(x) \approx mc^2 + \frac{1}{2}mc^2 x + \frac{3}{8}mc^2 \cdot x^2$ ← to order x^2 to order v^4

So $E(v) \approx mc^2 + \frac{1}{2}mc^2 \cdot \frac{v^2}{c^2} + \frac{3}{8}mc^2 \frac{v^4}{c^4} = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m \frac{v^4}{c^2}$

2. New expansions from old

Near $u=0$: $\frac{1}{1-u} = 1+u+u^2+u^3+u^4\cdots$ $\exp u = 1+\frac{1}{1!}u+\frac{1}{2!}u^2+\frac{1}{3!}u^3+\frac{1}{4!}u^4+\cdots$

$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \cdots$

(8) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x=0$.

To 3rd order, $\sin x \approx x - \frac{1}{6}x^3$ (memorise!)

then $(1+x)\sin x \approx (1+x)(x-\frac{1}{6}x^3)$

$\approx x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4$

$\approx x + x^2 - \frac{1}{6}x^3$ Correct to 3rd order

Can add, multiply expansions, preferred way to get expansions

(9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

$e^u \approx 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \frac{1}{24}u^4 + \dots$ so $e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}$

$\frac{1}{1-v} \approx 1 + v + v^2 + \dots$

$\frac{1}{1-(-x^3)} \approx 1 + (-x^3) + (-x^3)^2$

to 8th order.

Now subtract.