13. Differential Equations (27/11/2024)

Goals.

(1) Differential equations
  (a) What is a differential equation?
  (b) What is a solution to a differential equation?
  (c) Plugging ansatze into equations
(2) First order Linear DE

Last Time. Newton's method

Problem: Given f find x s.t. $f(x)=0$.

Idea: given guess x_n improve it to $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

(by linear approx)

(also discussed bisection)

ODE

Newton's goal of mechanics is to predict future. Unknown: $y(t)$, position of a particle as a function of time. Equation: involve derivatives

$F = ma$ means $\boxed{y''(t) = \frac{F(y)}{m}}$.

Math 100A – WORKSHEET 13
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

can check if 2,3 is a solution → *without* solving equation

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

$3^2 = 9 \neq 4$ ✗ $\qquad$ $3^2 = 9 = 3\cdot 3$ ✓

$2^2 = 4 = 4$ ✓ $\qquad$ $2^2 = 4 \neq 6 = 3\cdot 2$ ✗

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

$\frac{d(x^2)}{dx} = 2x \neq x^2$ ← inequality of functions

$\frac{d(e^x)}{dx} = e^x$ ✓ ← equality of functions

$\left(\frac{d(e^x)}{dx}\right)^2 = e^{2x} \neq 4e^x$

$\left(\frac{d(x^2)}{dx}\right)^2 = 4x^2 = 4(x^2)$ ✓

(3) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

$\frac{d(t^2)}{dt} + t^2 - 1 = 2t + t^2 - 1 \neq t^2$, $\frac{d(t+1)^2}{dt} + t^2 - 1 = 2(t+1) + t^2 - 1$

$= t^2 + 2t + 1$ ✓

Exponential

Equation: $y' = ry$ (y unknown, r constant)

Know: $\frac{d(e^t)}{dt} = e^t$ notice: $\frac{d(e^{rt})}{dt} = re^{rt} = r(e^{rt})$

so $y = e^{rt}$ is a solution

notice: for any constant C, $\frac{d(Ce^{rt})}{dt} = Cre^{rt} = r(Ce^{rt})$

so $\boxed{y = Ce^{rt}}$ are solutions

arbitrary constant (C) from equation (r)

Fact: this is the general solution (the list of all solutions)

Occurs in population growth models, radioactive decay

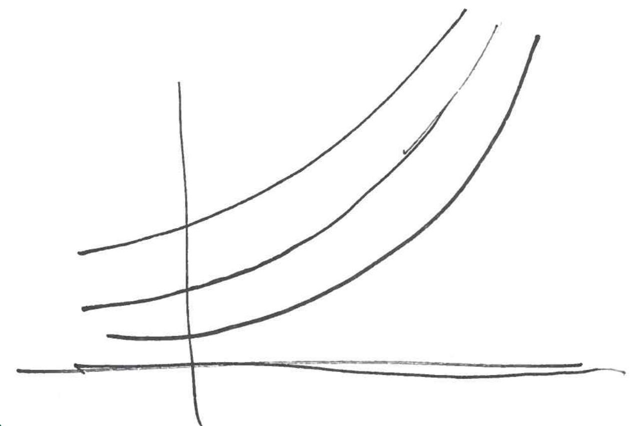
$r > 0$ get growth $r < 0$ get decay

"general solution"

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

Solutions are $y(t) = Ce^{1.04t}$

(maybe $C = 0$)



To make $y(0) = 100$, need $C \cdot e^{1.04 \cdot 0} = 100$ so $\boxed{C = 100}$

"initial condition"

equation for C.

so $\boxed{y(t) = 100e^{1.04t}}$ ← "particular solution"

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

Here

$$\frac{dw}{dx} = \frac{\frac{dy}{dx} \cdot z - y\frac{dz}{dx}}{z^2} = \frac{ayz - ybz}{z^2} = (a-b)\frac{y}{z} = (a-b)w$$

so $$\frac{d(w)}{d\cancel{x}} = (a-b)w$$

"Ansatz" = (intelligent) guess

2. Solutions by massaging and Ansätze

(7) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

If $y(t) = \sin(\omega t)$, $\frac{dy}{dt} = \omega\cos(\omega t)$, $\frac{d^2y}{dt^2} = -\omega^2 \sin(\omega t)$

so $y'' + 4y = (4-\omega^2)\sin(\omega t)$

See: If $\omega = 2$, get a solution $y = \sin(2t)$

($\omega = -2$, $\omega = 0$ also work)

Aside: $C\sin(2t)$ also works, $\sin(-2t) = (-1)\cdot\sin(2t)$

General solution is $C\sin(2t) + D\cos(2t)$

change of variable

(9) Consider the equation $\frac{dy}{dt} = a(y-b)$. ✓

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by u?

$$\frac{du}{dt} = \frac{dy}{dt} = a(y-b) = au$$

$$\boxed{\begin{array}{l}\text{solution to} \\ \quad y' = ay + d \\ \text{is } Ce^{at} - d/a\end{array}}$$

(b) What is the general solution for $u(t)$?

So $u(t) = C \cdot e^{at}$, C arbitrary

(c) What is the general solution for $y(t)$?

$y(t) = b + u(t) = b + Ce^{at}$.

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?

As $t \to \infty$, $e^{at} \to 0$, $y(t) \to b$

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?

At $t=0$, $y(0) = b + C$, so $C = y(0) - b$,

$$y(t) = b + (y(0) - b)e^{at}.$$

(10) Example: *Newton's law of cooling.* Suppose we place an object of temperature $T(0)$ in an environment of temperature T_{env}. It turns out that a good model for the temperature $T(t)$ of the object at time t is

$$\frac{dT}{dt} = -k\,(T - T_{\text{env}})$$

where $k > 0$ is a positive constant.

(a) Suppose $T(t) > T_{\text{env}}$. Is $T'(t)$ positive or negative? What if $T(t) < T_{\text{env}}$? Explain this in words.

if $T(t) > T_{env}$, $T - T_{env} > 0$ so $T'(t) = -k(T - T_{env}) < 0$

if $T(t) < T_{env}$, $T - T_{env} < 0$, $T'(t) > 0$

(b) A body is found at 1:30am and its temperature is measured to be 32.5°C. At 2:30am its temperature is found to be 30.3°C. The temperature of the room in which the body was found is measured to be 20°C and we have no reason to believe the ambient temperature has changed. What was the time of death?

Let $T(t)$ be the temperature t hours after 1:30am

Let $u(t) = T(t) - 20°C$

Then $\frac{du}{dt} = -ku$ for some k, so $u(t) = Ce^{-kt}$

Know: $u(0) = 32.5 - 20 = 12.5$, $u(1) = 10.3$

so $C = 12.5$, $10.3 = u(1) = 12.5 \cdot e^{-k}$

$\Rightarrow \boxed{k = \log \frac{12.5}{10.3}}$ ← sanity check $k > 0$.

① find t where $T(t) = 37°$, $u(t) = 17°$.

want: $12.5 \cdot e^{-\log \frac{12.5}{10.3} t} = 17°$ so $\boxed{t = -\frac{\log 17/12.5}{\log 12.5/10.3}}$ hours before 1:30am