14. The Euler Scheme (4/12/2024)

Final exam Dec 17^{th}

Advice on section page

Office hours continue

Goals.

(1) Review ODE; *derive* an ODE
(2) Solving an ODE numerically
 (a) The power of linear approximation
 (b) Calculating by hand
 (c) Calculating on a computer

Last Time. ODE

Diff eqn: Equation where (1) unknown function (2) involves derivatives

Eg.: $y' = f(y;x)$ for unknown $y = y(x)$

Ideas: (1) Testing if a formula is the formula of a solution
(2) Ansatz
(3) Particular solutions if our solution depends on a parameter, can determine the value of parameter from info s.t. $y(0)$, $y'(0)$...

1. Compound interest (Bernoulli 1683)

(1) Suppose you have a \$100 bank balance which earns an annual interest rate of 30%.

(a) Suppose the interest is paid once, at the end of the year. How much would your balance be at that time?

\$100 + 30% · \$100 = \$130

(b) Suppose instead that interest is paid four times a year. What is the quarterly interest *rate*? What would the balance be at the end of the first quarter?

per quarter: $7.5\% = \frac{30\%}{4}$, so after 1 quarter \$100 + 7.5% · \$100 = 107.5

Date: 4/12/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) Suppose further that interest is *compounded*: after every quarter the interest is added to the balance. What would be the balance at the end of the year?

After 2^{nd} quarter: \$107.5 + 7.5% \$107.5 ← key idea

Here: $\$107.5 \cdot (1 + 7.5\%) = 1.075 \times \$107.5$

so at end of year $\$100 \cdot (1.075)^4$.

(d) Suppose instead that interest is compounded *daily* and that at a particular day the balance is y dollars. What is the balance the next day?

daily rate: $\frac{0.30}{365}$, daily interest: $\frac{0.30}{365} \cdot y$

new balance: $y + \frac{0.30}{365} y$

(2) Suppose interest is compounded *continuously* and that at a particular time y the balance is $y(t)$ dollars, where t is measured in years.

(a) What is the approximate interest rate for the period between times $t, t+h$ if h is very small?

rate is: $\simeq \frac{h}{1\,\text{year}} \cdot \overset{0.30}{\cancel{30\%}} = 0{,}30\,h$

(b) What is the balance at time $t+h$?

$y(t+h) \simeq y(t) + y(t) \cdot .30 \cdot h = y(t) + (0{,}30\, y(t)) \cdot h$

so $\boxed{y'(t) = 0.3 \cdot y(t)}$

2. The Euler scheme

(3) Consider the ODE $y' = 0.3y$ from the previous page. We will work on the interval $[0, 1]$ with $y(0) = 100$.

(a) $[n = 1]$ What is $y'(0)$? Approximate $y(1)$ using a linear approximation.

$y'(0) = 0.3 \cdot y(0) = 30$

so $y(1) \approx y(0) + 30(1-0) = 100 + 30 = 130$

(b) $[n = 2]$ Approximate $y\left(\frac{1}{2}\right)$ using a linear approximation. What is $y'\left(\frac{1}{2}\right)$ approximately? Use this to estaimte $y(1)$.

approximate value to plug in

$y(\tfrac{1}{2}) \approx y(0) + 30 \cdot \tfrac{1}{2} = 115$ so $y'(\tfrac{1}{2}) \overset{\text{ODE}}{=} 0.3 \cdot y(\tfrac{1}{2}) \approx 0.3 \cdot 115 = 34.5$

so $y(1) \approx y(\tfrac{1}{2}) + y'(\tfrac{1}{2}) \cdot (1 - \tfrac{1}{2}) \approx 115 + 34.5 \cdot \tfrac{1}{2} = 132.25$

linear approx

approx values from previous step

(c) $[n = 3]$ do the same but dividing the interval into three steps.

$y(0) = 100$ so $y'(0) = 30$, $y(\tfrac{1}{3}) \approx 100 + 30 \cdot \tfrac{1}{3} = 110$

$\approx y(\tfrac{1}{3})$ $\approx y'(\tfrac{1}{3})$ $\tfrac{2}{3} - \tfrac{1}{3}$

so $y'(\tfrac{1}{3}) = 0.3 \cdot y(\tfrac{1}{3}) \approx 33$ so $y(\tfrac{2}{3}) \approx 110 + 33 \cdot \tfrac{1}{3} = 121$

so $y'(\tfrac{2}{3}) \approx 0.3 \cdot 121 = 36.3$ so $y(1) \approx 121 + 36.3 \cdot \tfrac{1}{3} = 133.1$

The Euler Scheme

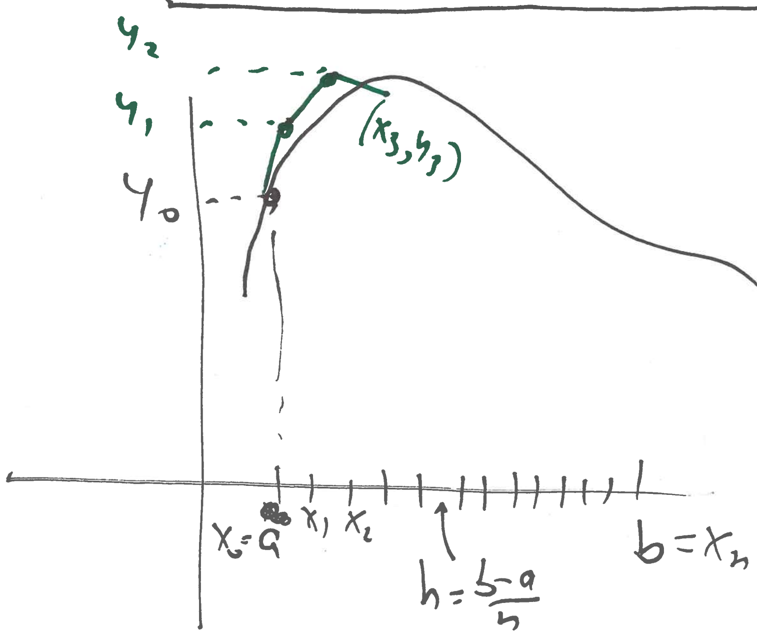
Input: ODE $y'(x) = f(y; x)$, interval $[a,b]$, initial value $y(a) = y_0$.

choose: n, number of steps $\Rightarrow$ step length $h = \Delta x = \frac{b-a}{n}$

Method: ~~step~~ set $x_0 = a,\ x_1 = a+h,\ x_2 = a+2h,\ \dots,\ x_n = a+nh = b$

Approximate: $y_1 = y_0 + f(y_0; x_0)h$ [our approx to $y(x_1)$]

$$\boxed{y_{i+1} = y_i + f(y_i; x_i)h}$$ [our approx to $y(x_{i+1})$ got from y_i]



(4) Consider the ODE $y' = x^3 - xy$ on the interval $[1, 3]$.

(a) Use two steps of the Euler scheme to approximate $y(3)$ if $y(1) = 0$.

$\Delta x = \frac{3-1}{2} = 1$; $y_0 = 0$

$x_0 = 1,\ x_1 = 2,\ x_2 = 3$

$y_1 = y_0 + f(y_0; x_0)h = \overset{y_0}{0} + (\overset{x_0^3}{1^3} - \overset{x_0}{1}\cdot\overset{y_0}{0})\cdot\overset{h}{1} = 1 \leftarrow y(2) \approx 1$

$y_2 = y_1 + f(y_1; x_1)h = \underset{y_1}{1} + (\underset{x_1^3}{2^3} - \underset{x_1}{2}\cdot\underset{y_1}{1})\cdot\underset{h}{1} = 7 \leftarrow y(3) \approx 7$

(b) For which A, B, C do we have that $y(x) = Ax^2 + B + Ce^{-x^2/2}$ satisfies the equation?