2. Limits.Asymptotes, Continuity (11/9/2024)

Goals.

(1) Limits of functions
(2) Existence and nonexistence of limits: blowup
(3) Asymptotes
(4) Continuity

MATX 1102 — in small class groups — Must be typeset, recommend LaTeX, can use overleaf.com. I use LyX.

News: ∘ Calculus Common Room ∘ Group project 1.

Last Time. Asymptotics

In a limit $x \to a$ (also $x \to \infty$, $-\infty$) can have $f(x) \ll g(x)$ meaning $\frac{g}{f} \to \infty$

can also have $f \sim g$ meaning $\frac{f}{g} \to 1$ [Can also have neither]

E.g.: As $x \to \infty$ $x^4 \gg x^2$, $x^2+1 \sim x^2$, as $x \to 0$, $x^4 \ll x^2$.

Math 100A – WORKSHEET 2
LIMITS. ASYMPTOTES, AND CONTIJUITY

(1) Review of asymptotics: analyze the expression $\frac{e^x + A\sin x}{e^x - x^2}$ as $x \to \infty$, $x \to 0$, $x \to -\infty$.

As $x \to \infty$, $e^x \gg A\sin x$ so $e^x + A\sin x \sim e^x$

growing bounded

$e^x \gg x^2$ so $e^x - x^2 \sim e^x$

$\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{e^x}{e^x} \sim 1$

As $x \to 0$, $e^x \sim 1$, $A\sin x \to 0$

$e^x \sim 1 \gg x^2$

so $\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{1}{1} \sim 1$

As $x \to -\infty$ $e^x - x^2 \sim (-x^2)$

but e^x, $A\sin x$ not comparable as $x \to -\infty$

so no simple asymptotic

1. Limits

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 5}(x^3 - x) = 5^3 - 5 = 120$

Date: 11/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

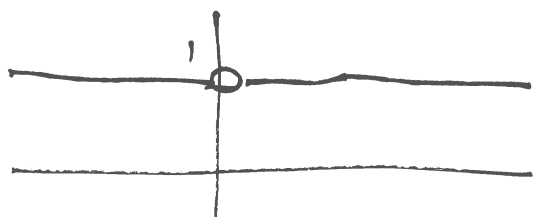
Limits

Informally: $\lim\limits_{x \to a} f(x) =$ "the value f would like to have at $x = a$".

Or: If we follow points $(x, f(x))$ along graph, where would we end as x approaches a?

Example: $\lim\limits_{x \to 0} \frac{x}{x} = \lim\limits_{x \to 0} 1 = 1$

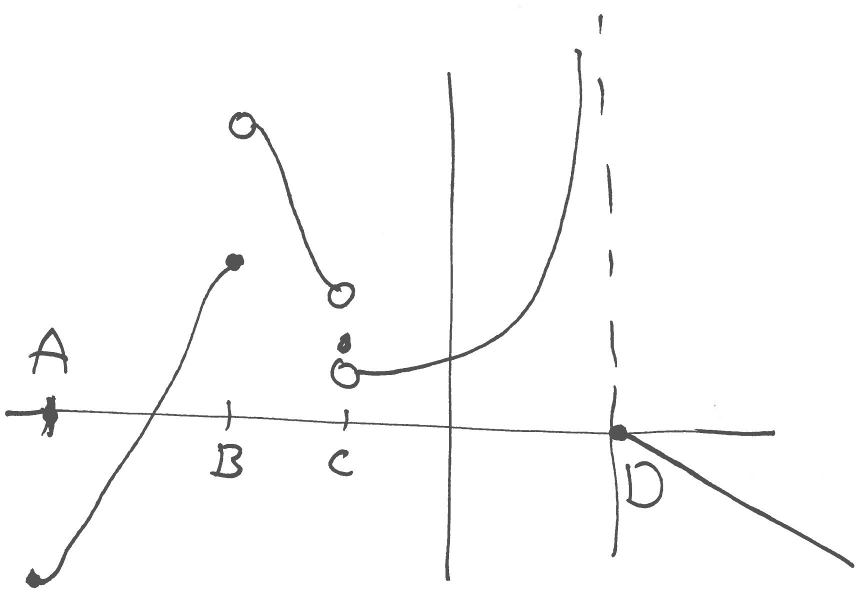
Graph:



$\lim\limits_{x \to a} f(x)$ determined by values near a, not at a.

(need f defined on both sides)

Example:



At B: $\lim_{x\to B^-} f(x) = f(B)$ | $\lim_{x\to B^+} f(x) =$ value on top

from left

so $\lim_{x\to B} f(x)$ does not exist ("DNE")

also say f "jumps" at B, has "jump discontinuity"

On the other hand $\frac{x}{x}$ has a "removable" discontinuity at $x=0$ (define function to be 1 there)

$\lim_{x\to D^-} f(x) = \infty$

means limit DNE, but is ∞ "in the extended sense".

(say f "blows up" at D or has "infinite discontinuity" or has a "vertical asymptote")

$\lim_{x\to D^+} f(x) = 0$

$\lim_{x\to 0} \frac{1}{|x|}$ DNE

but

$\lim_{x\to 0} \frac{1}{|x|} = \infty$ [in the extended sense]

Say f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

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f could be discontinuous at a if:

(1) it is undefined at a

(2) $\lim_{x \to a} f(x)$ DNE

(3) $\lim_{x \to a} f(x) \neq f(a)$

Promise: A function defined by formula is continuous where the formula makes sense

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$\Rightarrow$ If f is continuous then $\lim_{x \to a} f(x) = f(a)$

Remark on asymptotics

When $\left.\begin{matrix} f(x) \to \infty \\ \text{or } f(x) \to 0 \end{matrix}\right\}$, having $f \sim g$ gives

more details about how f grows/decays

When $f(x) \to L \neq 0$ just have $f \sim L$.

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$

here $x<1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2-x^2 = 2-1^2 = 1$

here $x>1$

$\Rightarrow \lim_{x\to 1} f(x) = 1$

(removable discontinuity at $x=1$)

(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

Again $\lim_{x\to 1^-} f(x) = 1 = f(1)$

Now $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4-x^2) = 3 \neq 1$ so $\lim_{x\to 1} f(x)$ DNE

(jump discontinuity at $x=1$)

$$\frac{3-3}{3^2+3-12} \text{ "=" } \frac{0}{0}$$

(3) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

Note if $x \neq 3$, $\frac{x-3}{x^2+x-12} = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4} \xrightarrow[x\to 3]{} \frac{1}{7}$

or: $\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+4)} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{7}$

(b) What about $\lim_{x\to -4} f(x)$?

This is $\lim_{x\to -4} \frac{1}{x+4}$. Since $x+4 \to 0$, $\frac{1}{x+4}$ is blowing up

If $x<-4$, $x+4<0$ so $\frac{1}{x+4}<0$, so $\lim_{x\to -4^-} \frac{1}{x+4} = -\infty$

If $x>-4$, $x+4>0$, so $\frac{1}{x+4}>0$, so $\lim_{x\to -4^+} \frac{1}{x+4} = +\infty$

($\lim_{x\to -4} f(x)$ DNE even in extended sense)

But $\lim_{y\to 0} \frac{1}{y^2} = +\infty$

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

As $x \to -3^+$, $x+2 \sim -1$ so $\frac{x+2}{x+3} \sim \frac{-1}{x+3}$ blows u[p] since $x+3 \to 0$. As $x \to -3^+$, $x+3 > 0$ so $-\frac{1}{x+3} < 0$ so $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$.

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$

(d) $\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2}$

Recap

Limits: "where the function is going".

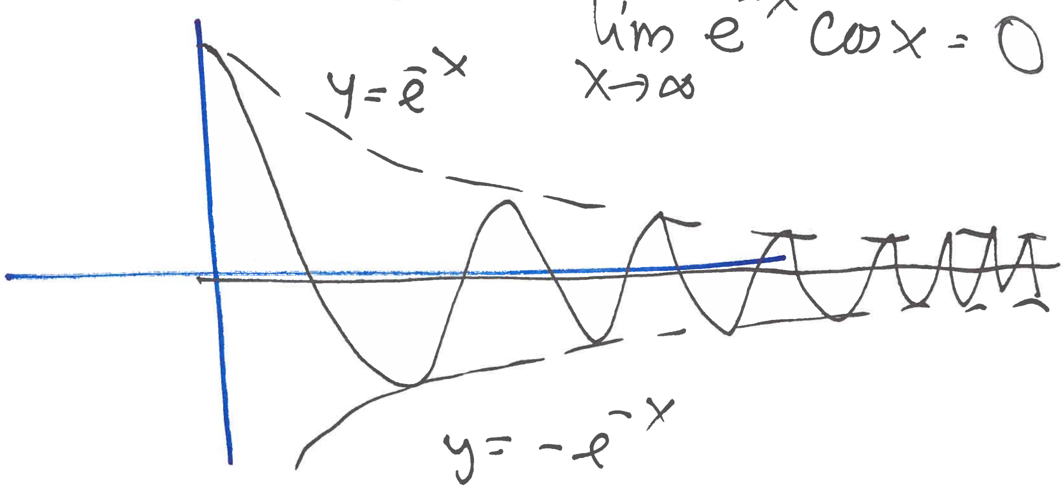
related to continuity.

Say f has a vertical asymptote at $x=a$ if it has an infinite one-sided limit there.

Say f has a horizontal asymptote $y=L$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

Example: $e^{-x}\cos x$ as $x\to\infty$

$$\lim_{x\to\infty} e^{-x}\cos x = 0$$



(b) (Final, 2015) $\frac{x+1}{x^2+2x-8}$

(7) (Quiz, 2015) Evaluate $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$

As $x \to -\infty$, $4x^2+x \sim 4x^2$ x is negative

So $\sqrt{4x^2+x} \sim \sqrt{4x^2} \stackrel{\downarrow}{=} -2x$

So $\frac{3x}{\sqrt{4x^2+x}-2x} \sim \frac{3x}{-2x-2x} \sim -\frac{3}{4}$

So $\lim\limits_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = -\frac{3}{4}$.