3. The Derivative (18/9/2024)

Goals.

(1) The derivative at a point
(2) Tangent lines & linear approximations
(3) The derivative as a function

Last Time. Limits

$\lim_{x\to a} f(x) = L$ encodes idea "as x approaches a, f(x) gets closer to L".

① Also makes sense as $x\to\infty$, $x\to-\infty$

② Can be extended to "$L=\infty$", "$L=-\infty$".

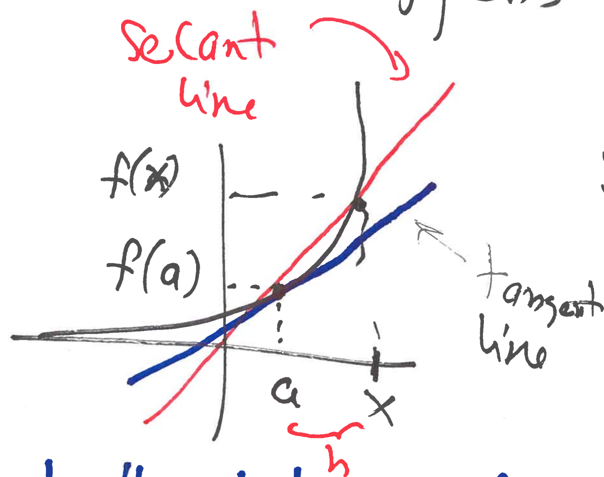
③ Allows us to interpret "limiting processes".

Derivatives

<u>Function</u>: gadget that maps "x values" to "y values".

<u>Goal</u>: understand what happens when x changes.

<u>Geometrically</u>.



start at $(a, f(a))$

want to understand $(x, f(x))$

Have line that "matches" function at $(a, f(a))$

(Call it the line <u>tangent</u> to graph/function)

To find this line, start with line through $(a, f(a))$, $(x, f(x))$

Second view

Start at $(a, f(a))$. Change x-value by h:

go to $x = a+h$ $\quad (h = x-a)$ ← the "small parameter"

Question: As $h \to 0$, what happens to $f(a+h)$?

Continuity: If $f(a+h) \xrightarrow[h\to 0]{} f(a)$

Next level: look at $f(a+h) - f(a)$ see how it decays

Typically:

$$f(a+h) - f(a) \sim ch$$

$$\Updownarrow$$

$$f(a+h) \approx f(a) + ch$$ ← "linear approx. to f about $x=a$"

Def: If this is true call c "the derivative of f at a"

write $f'(a)$, $\left.\frac{df}{dx}\right|_{x=a}$, ...

Equivalently:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

Math 100A – WORKSHEET 3
THE DERIVATIVE

1. Three views of the derivative

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2,4)$ is a point on the graph of $y = f(x)$.

(a) Let (x, x^2) be another point on the graph, close to $(2,4)$. What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?

slope $= \frac{x^2-4}{x-2} = x+2 \xrightarrow[x\to 2]{} 4.$

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2+h)$ as $h \to 0$? What about $f(2+h) - f(2)$?

$f(2+h) = (2+h)^2 = 4+4h+h^2 \sim 4 = f(2)$ as $h \to 0$

$f(2+h) - f(2) = (4+4h+h^2) - 4 = 4h + h^2 \sim \boxed{4}h$

(c) What is $\lim_{h\to 0} \frac{(2+h)^2-2^2}{h}$?

$\lim_{h\to 0} \frac{(2+h)^2-2^2}{h} = \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0}(4+h) = \boxed{4}$

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2,4)$?

Date: 18/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

It is $Y = 4(X-2) + 4$

slope

Ex: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$\approx x^3 + (3x^2)h$ as $h \to 0$

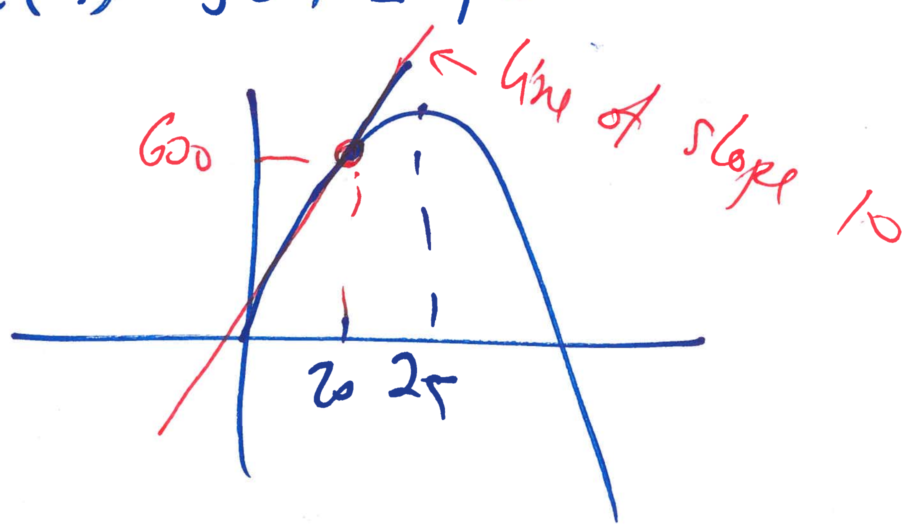
so x^3 is differentiable at x, derivative is $3x^2$.

Problem 2: $k(20+h) - k(20) \approx 10h$

so if $h > 0$, small, $k(20+h) > k(20)$

so increase temperature

Q: $k(T) = 50T - T^2$



2. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a+h) \approx f(a) + f'(a)h$

(3) Use a definition of the derivative to find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

$f(3+h) - f(3) = (3+h)^2 - 3^2 = 6h + h^2 \sim 6h$ as $h \to 0$

so $f'(3) = 6$.

(b) $f(x) = \frac{1}{x}$, any a.

$f(a+h) - f(a) = \frac{1}{a+h} - \frac{1}{a} = \frac{a-(a+h)}{a(a+h)} = -\frac{h}{a(a+h)} \sim -\frac{h}{a \cdot a}$

$= \left(-\frac{1}{a^2}\right) \cdot h$

so $f'(a) = -\frac{1}{a^2}$

The derivative function

Def: Say f is **differentiable** at $x=a$ if $f'(a)$ exists (e.g. if $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists)

Requirement: f must be continuous to be differentiable.

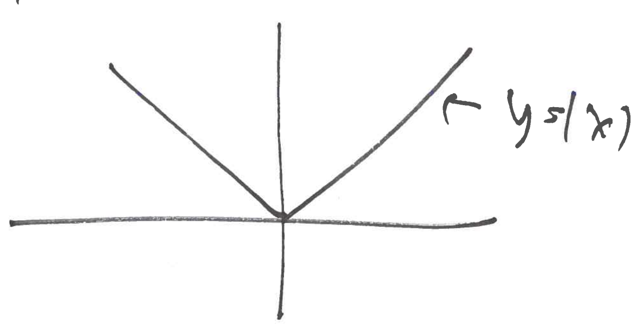
If f is diff. on an interval, we get the derivative function f', $\frac{df}{dx}$, $\frac{d}{dx}f$, Df, $D_x f$, -

such that

(the function s.t. $f'(x)$ is the derivative at x)

Example: $\frac{d(x^n)}{dx} = nx^{n-1}$

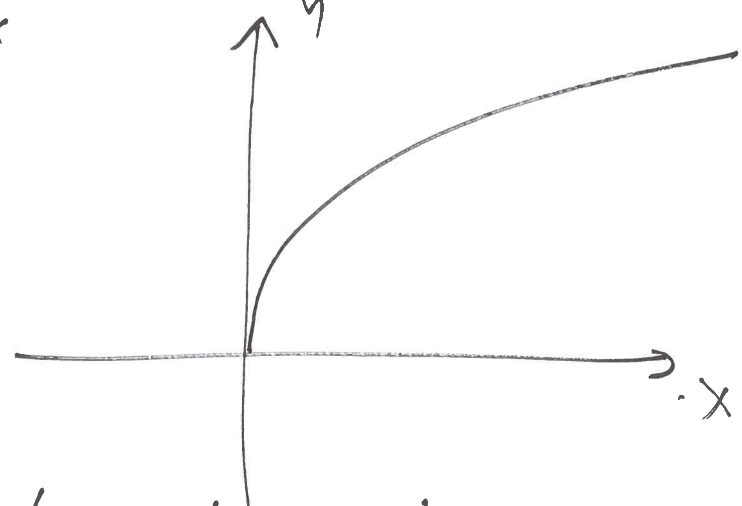
Example: $\frac{d(|x|)}{dx} = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$



Example: $y = \sqrt{x}$

$\Updownarrow$

$y^2 = x$



Tangent line at $(0,0)$ is vertical ("infinite slope")

so fcn is not diff. at $(0,0)$

(here $\frac{dy}{dx} = \frac{d(x^{1/2})}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$)

Once we have f' can differentiate again

get $(f')' = f''$, f''', ... also write $f^{(2)}, f^{(3)}, f^{(4)}, \ldots$

also $\frac{d^2}{dx^2}f$, $\frac{d^3 f}{dx^3}$, $\frac{d^4 f}{dx^4}$, ...

3. The tangent line

(6) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

Slope: $\left.\dfrac{d(\sqrt{x})}{dx}\right|_{x=4} = \left[\dfrac{1}{2\sqrt{x}}\right]_{x=4} = \dfrac{1}{4}$

Line: $Y = \frac{1}{4}(X - 4) + 2$

Or, $f'(x) = \dfrac{1}{2\sqrt{x}}$ so $f'(4) = \dfrac{1}{4}$

(7) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

at $x=1$ line passes through $(1, 6)$

has slope 4.

Now check for each function

(8) Find the lines of slope 3 tangent to the curve $y = x^3 + 4x^2 - 8x + 3$.

Say line is tanget to curve at $(x, f(x))$.

then $f'(x) = 3$

Solve this equation to get x values,

. . .

(9) The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

4. Linear approximation

Definition. $f(a+h) \approx f(a) + f'(a)h$

(10) Estimate

(a) $\sqrt{1.2}$

let $f(x) = \sqrt{x}$, work about $a=1$

$f(1) = \sqrt{1} = 1$, $f'(1) = \left[\frac{1}{2\sqrt{x}}\right]_{x=1} = \frac{1}{2}$

$f(x) \simeq 1 + \frac{1}{2}(x-1)$

$f(1+h) \simeq 1 + \frac{1}{2}h$

so $f(1.2) \simeq 1 + \frac{1}{2} \cdot 0.2 = 1.1$

(b) (Final, 2015) $\sqrt{8}$