4. Computing derivatives (27/9/2024)

Goals.

(1) Combining linear approximations
(2) Linearity of the derivative
(3) The product and quotient rules

Last Time. Linear Approximation + Defn of the Derivative

Idea: If we can approximate f near a in that

$f(x) \approx f(a) + c\cdot(x-a)$ ← preferred form

$\Leftrightarrow f(a+h) \approx f(a) + ch$ ← good for proofs

We call c the derivative of f at a, write $f'(a) = c$

Call $\left.\begin{array}{l} f(x) \approx f(a) + f'(a)(x-a) \\ f(a+h) \approx f(a) + f'(a)h \end{array}\right\}$ the linear approximation to f

Call $y = f(a) + f'(a)(x-a)$ the line tangent to $y = f(x)$ at $(a, f(a))$

Technical aside: "approx" means

$$f(x+h) - (f(x) + f'(a)h) \ll h$$

Math 100A – WORKSHEET 4
COMPUTING DERIVATIVES

1. Review of the derivative

(1) Expand $f(x+h)$ to linear order in h for the following functions and read the derivative off:

(a) $f(x) = bx$

$f(x+h) = b(x+h) = \underbrace{bx}_{f(x)} + bh$ so $f'(x) = b$

(b) $g(x) = ax^2$

correct to order h

$g(x+h) = a(x+h)^2 = ax^2 + 2axh + ah^2 \approx ax^2 + (2ax)h$

so $g'(x) = 2ax$

(c) $\cancel{h}H(x) = ax^2 + bx. = g(x) + f(x)$

$H(x+h) = g(x+h) + f(x+h)$

to linear order in h $\approx (g(x) + 2ax \cdot h) + (f(x) + bh)$

$\approx (g(x) + f(x)) + (2ax + b)h$

so $H'(x) = 2ax + b$

(d) $i(x) = \frac{1}{b+x}$

(e) $j(x) = 4x^4 + 5x$ (hint: use the known linear approximation to $2x^2$)

$2(x+h)^2 \approx 2x^2 + 4x \cdot h$ to linear order in h

So $4(x+h)^4 = \left(2(x+h)^2\right)^2 \approx \left(2x^2 + 4xh\right)^2$ ← Key new idea: replace function by approximation

$\approx 4x^4 + 16x^3h + 16x^2h^2$

$\approx 4x^4 + 16x^3h$

Also $5(x+h) \approx 5x + 5h$

So $j(x+h) \approx j(x) + (16x^3 + 5)h$ so $j'(x) = 16x^3 + 5$

More abstractly

Say $f(x+h) \approx f(x) + f'(x)h$

$g(x+h) \approx g(x) + g'(x)h$

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① $(f+g)(x+h) = f(x+h) + g(x+h) \approx$ ← to 1st order in h

$\approx (f(x) + f'(x)h) + (g(x) + g'(x)\cdot h)$

$\approx (f+g)(x) + (f'(x) + g'(x))h$

$\Rightarrow \boxed{(f+g)'(x) = f'(x) + g'(x)}$ ← sum rule

② $(af)(x+h) = a\cdot f(x+h) \approx a(f(x) + f'(x)h)$

$\approx (af)(x) + (a\cdot f'(x))\cdot h$

so $\boxed{(af)'(x) = a\cdot f'(x)}$

linearity of the derivative

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③ $(fg)(x+h) = f(x+h)\cdot g(x+h) \approx (f(x) + f'(x)h)\cdot(g(x) + g'(x)h)$ to 1st order in h

$\approx (fg)(x) + (f'(x)g(x) + f(x)g'(x))h + f'(x)g'(x)h^2$

$\approx (fg)(x) + (f'(x)g(x) + f(x)g'(x))h$

so $\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x)}$ ← product rule

$$\frac{f}{g}(x+h) \approx \frac{f(x)+f'(x)h}{g(x)+g'(x)h} \approx \left(f(x)+f'(x)h\right)\left(\frac{1}{g(x)}-\frac{g'(x)h}{g(x)^2}\right)$$

$$\approx \frac{f}{g}(x)+\left(\frac{f'(x)}{g(x)}-\frac{f(x)g'(x)}{g(x)^2}\right)h$$ (to 1st order)

$$\frac{1}{b+t} \approx \frac{1}{b}-\frac{t}{b^2}$$ as $t \to 0$ (see WS 3)

(use with $b=g(x)$ $t=g'(x)h$)

$$\Rightarrow \boxed{\left(\frac{f}{g}\right)'(x)=\frac{f'(x)}{g(x)}-\frac{f(x)g'(x)}{g(x)^2}=\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}}$$

(quotient rule)

2. Arithmetic of derivatives

(2) Differentiate

(a) $f(x) = 6x^\pi + 2x^e - x^{7/2}$

$$\frac{df}{dx} \underset{\uparrow}{=} 6\frac{d(x^\pi)}{dx} + 2\frac{d(x^e)}{dx} - \frac{d(x^{7/2})}{dx} \underset{\uparrow}{=} 6\pi x^{\pi-1} + 2e\cdot x^{e-1} - \frac{7}{2}x^{5/2}$$

Linearity

power law rule

(b) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

$$\frac{dg}{dx} \underset{\uparrow}{=} \frac{d(x^2)}{dx}\cdot e^x + x^2\frac{d(e^x)}{dx} \underset{\uparrow}{=} 2xe^x + x^2e^x$$

prdt rule

power low, exponential

$$\frac{d}{dx}(x^a e^x) = x^{a-1}(a+x)e^x.$$

(c) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$\frac{dh}{dx} \underset{\text{quot. rule}}{=} \frac{(x^2+3)'(2x-1)-(x^2+3)(2x-1)'}{(2x-1)^2} = \frac{2x(2x-1)-2(x^2+3)}{(2x-1)^2}$$

$$= \frac{2x^2-2x-6}{(2x-1)^2}$$

(d) $\frac{x^2+A}{\sqrt{x}}$

① use quotient rule ② write it as $(x^2+A)\cdot x^{-\frac{1}{2}}$ use prod rule

③ write as $x^{3/2} + Ax^{-\frac{1}{2}}$

so $\left(\frac{x^2+A}{\sqrt{x}}\right)' = \frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$.

(3) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

Strategy

① differentiate expression for f.

② plug 4 into expression
get expression involving A.

③ equate expression to $\frac{3}{16}$.

④ do algebra to get quadratic equation

① $f'(x) = \frac{(\sqrt{x}+A) - x\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$

② $f'(4) = \frac{1+A}{(2+A)^2}$ ③ $\frac{1+A}{(2+A)^2} = \frac{3}{16}$

④ $16+16A = 3(A^2+4A+4)$

So $\boxed{3A^2-4A-4=0}$

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.

$f(x) \approx f(1) + f'(1)(x-1) \approx 1 + 3(x-1)$

$g(x) \approx 2 + 4(x-1)$ ← about $x = 1$, to 1^{st} order.

So $(fg)(x) \approx (1+3(x-1))(2+4(x-1))$

$\approx 2 + 10(x-1) + 12(x-1)^2 \approx 2 + 10(x-1)$

to 1^{st} order in $(x-1)$

(b) Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

By pdt rule $(fg)'(1) = f'(1)\cdot g(1) + f(1)g'(1)$

$= 3 \cdot 2 + 1 \cdot 4 = 10$

By quot. rule $(f/g)'(1) = \dfrac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \dfrac{3\cdot 2 - 1\cdot 4}{2^2} =$

note: diff rules are pointwise statements, not only about expressions

(5) Evaluate

(a) $(x \cdot x)'$ and $(x') \cdot (x')$. What did we learn?

(b) $\left(\frac{x}{x}\right)'$ and $\frac{(x')}{(x')}$. What did we learn?

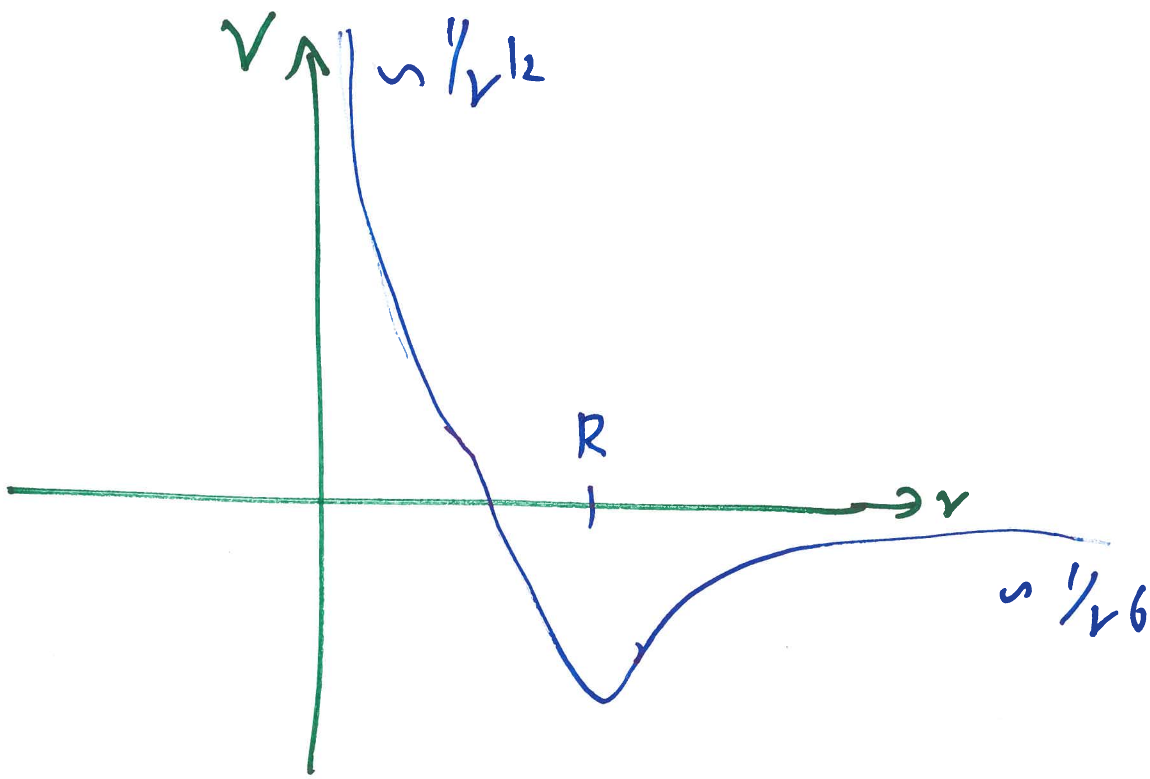
(6) The *Lennart–Jones potential* $V(r) = \epsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right)$ models the electrostatic potential energy of a diatomic molecule. Here $r > 0$ is the distance between the atoms and $\epsilon, R > 0$ are constants.

(a) What are the asymptotics of $V(r)$ as $r \to 0$ and as $r \to \infty$?

As $r \to 0$, $V(r) \sim \dfrac{\varepsilon R^{12}}{r^{12}}$

As $r \to \infty$, $V(r) \sim -2\varepsilon R^6/r^6$

(b) Sketch a plot of $V(r)$.



(c) Find the derivative $\frac{dV}{dr}(r) =$

$$\frac{dV}{dr} = \varepsilon \cdot (-12) \frac{R^{12}}{r^{13}} + 12\varepsilon \frac{R^6}{r^7} =$$

$$= 12\varepsilon \frac{R^6}{r^7}\left(1 - \frac{R^6}{r^6}\right)$$

(d) Where is $V(r)$ increasing? decreasing? Find its minimum location and value.

So $V(r)$ inc/dec as $\frac{R^6}{r^6} < 1$, $\frac{R^6}{r^6} > 1$

inc as $r^6 > R^6$, dec as $r^6 < R^6$

$r > R$ as $r < R$

So decrease on $(0, R)$, inc on (R, ∞)

minimum at $r = R$.