5. The Chain rule (2/10/2024)

Next week: midterm (60 minutes) see Canvas for material & policies

Practice exam: use under exam conditions

Goals.
(1) The Chain Rule: theory
(2) The Chain Rule: examples
(3) Logarithmic differentiation

Last Time. (1) Product & Quotient rules (2) combining linear approximations

(3) (trig functions)

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

(1) pointwise statements

(2) explained via linear approximation

Math 100A – WORKSHEET 5
THE CHAIN RULE

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.

(a) Expand $\sin(y+k)$ to first order in k. Write down the linear approximation to $\sin y$ about $y = a$.

$\sin(y+k) \approx \sin y + (\cos y)k$ | $\sin y \approx \sin a + (\cos a)\cdot(y-a)$

(b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

$F(x+h) = \sin(3(x+h)) = \sin(\underbrace{3x}_{y} + \underbrace{3h}_{k}) \approx \sin(3x) + \cos(3x)\cdot 3h$

unwinding definition

$\approx F(x) + (\cos(3x)\cdot 3)h$

so $F'(x) = 3\cos(3x)$

Date: 2/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

In General

Given functions $x \to g(x)$, $y \to f(y)$, their composition is the function $(f \circ g)(x) = f(g(x))$

Rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ ("Chain rule")

Or: if $y = g(x)$, $z = f(y)$ have

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

In the example have $y = 3x$, $z = \sin y$

so $\frac{dz}{dx} = \frac{d(\sin y)}{dy} \cdot \frac{dy}{dx} = (\cos y) \cdot 3 = \cos(3x)$.

Indeed, if

$$g(x+h) \approx g(x) + g'(x)h$$
$$f(y+k) \approx f(y) + f'(y)k$$

Then

$$f(g(x+h)) \approx f(\underbrace{g(x)}_{y} + \underbrace{g'(x)h}_{k}) \approx f(g(x)) + f'(g(x)) \cdot g'(x)h$$

so $(f \circ g)(x+h) \approx (f \circ g)(x) + (f'(g(x)) \cdot g'(x)) \cdot h$

(2) Write each function as a composition and differentiate

(a) e^{3x}

Let $f(y) = e^y$, Then $e^{3x} = f(g(x))$ and
$g(x) = 3x$

$$\frac{d(e^{3x})}{dx} = e^y \cdot 3 = e^{3x} \cdot 3$$

(b) $\sqrt{2x+1}$

~~take~~ ~~[illegible]~~ let $y = 2x+1$ Then $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$
$z = \sqrt{y}$

$$= \frac{1}{2\sqrt{y}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$$

(c) (Final, 2015) $\sin(x^2)$

let $f(\theta) = \sin\theta$
with $\theta = x^2$ have $\frac{d(\sin(x^2))}{dx} = \frac{d\sin\theta}{d\theta} \cdot \frac{d\theta}{dx}$

$$= \cos(x^2) \cdot 2x$$

(d) $(7x + \cos x)^n$.

This is y^n where $y = 7x + \cos x$

So $$\frac{d(7x+\cos x)^n}{dx} \overset{\text{chain rule}}{=} \frac{d(7x+\cos x)^n}{d(7x+\cos x)} \cdot \frac{d(7x+\cos x)}{dx} =$$

$$= \underbrace{n(7x+\cos x)^{n-1}}_{\text{power law rule}} \underbrace{(7-\sin x)}_{\text{linearity}}$$

(3) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

Differentiate f. By the chain rule,

$$f'(x) = g'(2\sin x) \cdot 2\cos x$$

$$f'\left(\tfrac{\pi}{4}\right) = g'\left(2\sin\tfrac{\pi}{4}\right) \cdot 2\cos\tfrac{\pi}{4}$$

$$= g'\left(2 \cdot \tfrac{1}{\sqrt{2}}\right) \cdot 2\,\tfrac{1}{\sqrt{2}} = g'(\sqrt{2}) \cdot \sqrt{2}$$

$$= \sqrt{2} \cdot \sqrt{2} = 2$$

Error 1: "I don't know what $g(y)$ is so can't diff".

Error 2: $f'(x) \neq g'(2\sin x)$ (forgot chain rule)

(4) Differentiate

(a) a^x for fixed $a > 0$ (hint: $a = e^{\log a}$)

If $a = e^{\log a}$, $a^x = (e^{\log a})^x = e^{(\log a)\cdot x}$

So $\frac{d}{dx} e^{(\log a)\cdot x} = e^{(\log a)\cdot x} \cdot (\log a) = a^x \log a$

(b) $7x + \cos(x^n)$

$$\frac{d(7x+\cos(x^n))}{dx} = 7 + \frac{d}{dx}(\cos(x^n)) = 7 - \sin(x^n) n x^{n-1}.$$

linearity

chain rule

(c) $e^{\sqrt{\cos x}}$

$$\left(e^{\sqrt{\cos x}}\right)' = e^{\sqrt{\cos x}} \frac{1}{2\sqrt{\cos x}} (-\sin x)$$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Diff both sides, get equation

$$f'(g(x)) \cdot g'(x) = 3x^2$$

so $f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$

so $g'(4) = \frac{48}{5}$.

Logarithms

Say $y = \log x$, want $\frac{dy}{dx}$.

This the same as $x = e^y$

Then $$1 = \frac{dx}{dx} = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \cdot \frac{dy}{dx}$$

chain rule

$$\Rightarrow \quad 1 = e^y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{d(\log x)}{dx} = \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

2. Logarithmic differentiation

(6) $\log\left(e^{10}\right) = 10$ $\log(2^{100}) = 100 \log 2$

(7) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} = \frac{1}{ax} \cdot a = \frac{1}{x}$

or: $\log(ax) = \log a + \log x$

$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) = \frac{1}{t^2+3t} \cdot (2t+3)$

$= \frac{2t+3}{t^2+3t}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$

$= 2x \log(1+x^2) + x^2 \frac{2x}{1+x^2}$

$= 2x \log(1+x^2) + \frac{2x^3}{1+x^2}$

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$

(8) (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

Take $\log y = \log(x^2+1) + \log(\sin x) - \frac{1}{2}\log(x^3+3) + \cos x$

Diff both sides wrt x Get:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$$

So $$\frac{dy}{dx} = (x^2+1)\sin x \frac{1}{\sqrt{x^3+3}} e^{\cos x}\left[\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x\right]$$

multiply by y,
use defn of y.

(9) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) x^n

$$\frac{d(x^n)}{dx} = x^n \cdot \frac{d(\log(x^n))}{dx} = x^n \frac{d(n \log x)}{dx} = x^n \cdot \frac{n}{x} = nx^{n-1}$$

(b) x^x

$$(x^x)' = x^x \cdot (\log(x^x))' = x^x(x \cdot \log x)' = x^x\left(\log x + \frac{x}{x}\right)$$

$$= x^x(\log x + 1)$$

(c) $(\log x)^{\cos x}$

$$\frac{d}{dx}\left((\log x)^{\cos x}\right) = (\log x)^{\cos x}\frac{d}{dx}\left(\log\left((\log x)^{\cos x}\right)\right)$$

$$= (\log x)^{\cos x} \cdot \frac{d}{dx}\left(\cos x \cdot \log(\log x)\right)$$

$$= (\log x)^{\cos x}\left[-\sin x \log\log x + \frac{\cos x}{\log x \cdot x}\right]$$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.