7. Related Rates (16/10/2024)

Goals.

(1) Direct application of the chain rule
(2) Problem-solving

Last Time. Inverse trig

(1) to define inverse fcn (often) need to restrict domain:

$\theta = \arcsin x$ if $\sin\theta = x$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

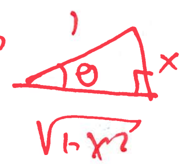
$\theta = \arccos x$ if $\cos\theta = x$, $\theta \in [0, \pi]$

$\theta = \arctan x$ if $\tan\theta = x$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\left(\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}.\right)$

(2) to evaluate trig(arc trig x):

draw triangle, angle θ, set one side $= x$, one side $= 1$
Calculate third side by Pythagoras, evaluate

eg. $\cos(\arcsin x) = \sqrt{1-x^2}$



(3) Set $\theta = \arcsin x$, $\sin\theta = x$, take $\frac{d}{dx}$ get $1 = \cos\theta \frac{d\theta}{dx} \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Related Rates Summary

(0) Read problem: understand the idea, draw a picture if possible.
(1) Assign names:
- Choose axes, quantities of interest.
- Give a *name* to each quantity of interest.

(2) Function: determine the *relations* between the quantities of interest, ending up with a relation between just two.
(3) Calculus: differentiate the relation using the chain rule
(4) Interpretation: solve the problem using the calculus result.
- Make *sanity checks* (area can't be negative, for example).

Basic idea: Have relation (identity) involving some variables
If they all depend on some independent variable
Can differentiate the relation to get new relation
involving derivatives.

Math 100A – WORKSHEET 7
APPLICATIONS OF THE CHAIN RULE

1. Related Rates 1: differentiation

(1) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $(1, \sqrt{3})$ we have $\frac{dy}{dt} = 1$. Find $\frac{dx}{dt}$.

Diff wrt t we get: $2y\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 2\frac{dx}{dt} = (3x^2+2)\frac{dx}{dt}$

at given time

so $\frac{dx}{dt} = \frac{2y}{3x^2+2}\frac{dy}{dt} = \frac{2\sqrt{3}}{5} \cdot 1 = \frac{2\sqrt{3}}{5}$.

(2) Air is pumped into a spherical balloon at the rate of $13\text{cm}^3/\text{s}$. How fast is the radius of the balloon changing when it is 15cm?

Let r be the radius of the balloon. The volume of the air is then $V = \frac{4}{3}\pi r^3$. Then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

so $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{13}{4\pi \cdot (15)^2} = \frac{13}{900\pi} \frac{\text{cm}}{\text{sec}}$

Date: 16/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(4) The *ideal gas law* provides that $PV = NkT$ for mass of N particles of an ideal gas, where P is the pressure, V is the volume, T is the temperature, and k is Boltzmann's constant.

(a) Suppose that we heat a mass of gas in a container of fixed volume. Relate the rate of change of the pressure to the rate of change of the temerature.

(b) Suppose that we compress a piston while holding the temperature constant. Relate the rate of change of the pressure of the gas to the rate of change of its volume. Is the pressure increasing or decreasing?

(3) If we place an object at distance p from a thin lens, the *lenscrafter's formula* provides that the distance of the image from the lens, q, is determined by

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f},$$

where f is the *focal length* of the ~~length~~ lens. Suppose that $f = 10$cm and $p = 30$cm. If we move the object away from the lens at the rate of 4cm/s, how fast is the image moving? In which direction is it moving?

Calculus step

$$-p^{-2}\frac{dp}{dt} - q^{-2}\frac{dq}{dt} = 0$$

given: p, $\frac{dp}{dt}$

want: $\frac{dq}{dt}$

get q from $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$

At given time, $p = 30$cm

$f = 10$cm

$\frac{dp}{dt} = 4\frac{\text{cm}}{\text{sec}}$

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{1}{10} - \frac{1}{30} = \frac{1}{15}$$ So $q = 15$ cm

So $$\frac{dq}{dt} = -\frac{q^2}{p^2}\frac{dp}{dt} = -\frac{15^2}{30^2}\cdot 4 = -1\frac{\text{cm}}{\text{sec}}$$

The image is moving at $1\frac{\text{cm}}{\text{sec}}$ toward the lens.

answering the question

(5) A closed rectangular box has sides of lengths 4, 5, 6cm. Suppose that the first and second sides are lengthening by $2\frac{\text{cm}}{\text{sec}}$ while the third side is shortening by $3\frac{\text{cm}}{\text{sec}}$.

(a) How fast is the volume changing?

Let the sides be x, y, z in order. Then the volume is $V = xyz$. Thus $\frac{dV}{dt} = \frac{dx}{dt}yz + x\frac{dy}{dt}z + xy\frac{dz}{dt}$

product rule

At the given time,

$$\frac{dV}{dt} = 2\cdot 5\cdot 6 + 4\cdot 2\cdot 6 + 4\cdot 5\cdot(-3) = 48\,\frac{\text{cm}^3}{\text{sec}}.$$

(b) How fast is the surface area changing?

$(A = 2xy + 2yz + 2zx)$

(c) How fast is the main diagonal changing?

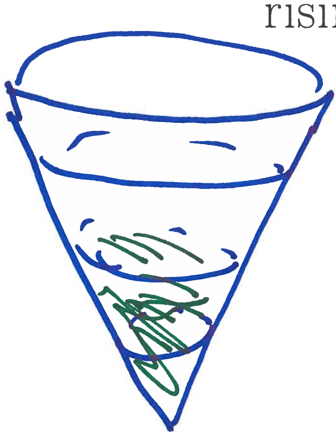
The length is $D = \sqrt{x^2+y^2+z^2}$, or $D^2 = x^2+y^2+z^2$.

(at given time $D = \sqrt{77}$) . . –

2. Related Rates 2: problem-solving

(6) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

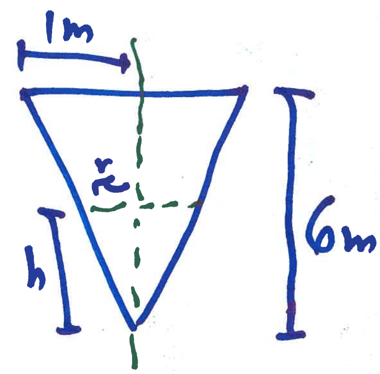


let V = volume of water
h = height of water
r = radius of top of water

names

Then water is a cone so $V = \frac{1}{3}\pi r^2 h$ relation

Draw a vertical cross-section:



from similar triangles get $\frac{r}{h} = \frac{1}{6}$

~~(relation~~

$\Rightarrow r = \frac{1}{6}h$

relation

so $V = \frac{1}{3}\pi \cdot \frac{1}{6^2} h^3$

Diff wrt t: $\frac{dV}{dt} = \frac{\pi}{36} h^2 \frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{36}{\pi h^2}\frac{dV}{dt}$

calculus

$= \frac{36}{\pi \cdot 5^2} \cdot 5 = \frac{36}{5\pi} \frac{\text{m}}{\text{min}}$

at given time

answer

(b) The drain is unclogged and water begins to drain at the rate of $(5 + \frac{\pi}{4})\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Still have $\frac{dV}{dt} = \frac{\pi}{36} h^2 \frac{dh}{dt}$ Now $\frac{dh}{dt} = -1 \frac{m}{min}$

$\frac{dV}{dt} = -\frac{\pi}{4} \frac{m^3}{min}$

so at given time $h^2 = \frac{36}{\pi} \cdot \left(-\frac{\pi}{4}\right) / (-1) = 9 \, m^2$

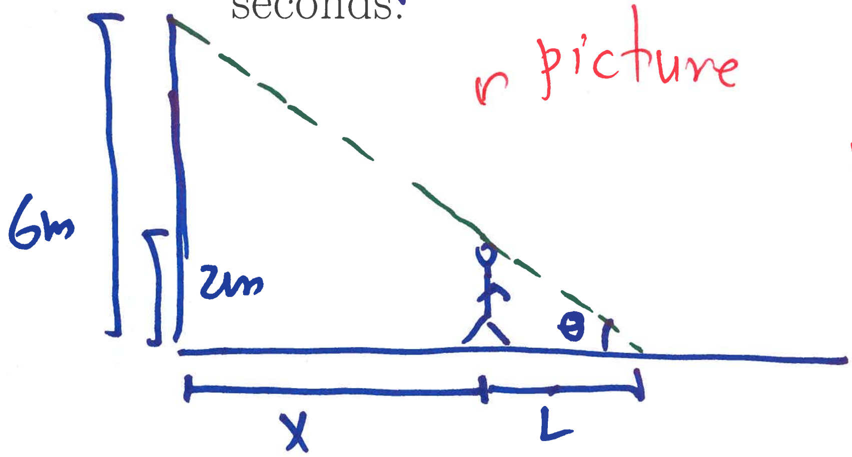
so $h = 3m.$ $(h > 0)$

(so $r = \frac{h}{6} = \frac{1}{2} m$, so $V = \frac{1}{3} \pi \left(\frac{1}{2}\right)^2 \cdot 3 = \frac{\pi}{4} m^3$)

(aside to show we can have more work to get an answer)

(c) Repeat the problem with tank upside-down (vertex on top).

(7) (Final, 2019) A 2m tall woman is running at night, moving away from a 6m-tall lamp-post. Her velocity t seconds after leaving the lamp-post is given (in metres per second) by $v(t) = 4 - \sin(2\pi t)$. How quickly is the length of her shado~~s~~w changing after 3 seconds?



← picture

names: Let x be the distance to the lamp, L the length of the shadow

given: $\frac{dx}{dt} = v(t) = \ldots$

By similar triangles

$$\frac{1}{\tan\theta} = \frac{L}{2m} = \frac{L+x}{6m} \quad \text{geometry}$$

relation

$$\Downarrow \quad 3L = L + x \Rightarrow L = \frac{1}{2}x$$

Thus $\frac{dL}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}v(t) = \frac{1}{2}(4 - \sin(2\pi t))$ ← calculus

At $t=3$, $\sin(2\pi t) = \sin(6\pi) = 0$ so $\frac{dL}{dt} = \frac{1}{2}4 = 2\frac{m}{sec}$.

answer