8. Curve sketching (23/10/2024)

Goals.
(1) Convexity
(2) Curve sketching

MT1 (0) See "Tests" page on Canvas
(1) Talk to instructors
(2) ~~[illegible]~~ esp before regrade

Last Time. "Related rates" = chain rule + problem-solving/setup

Example: from physics, $PV = NkT$ (N, k constants)

1(a) If V constant, $V\frac{dP}{dt} = Nk\frac{dT}{dt} \Rightarrow \frac{dP}{dt} = \frac{Nk}{V}\frac{dT}{dt}$

(b) If T constant, $\frac{dP}{dt}V + P\frac{dV}{dt} = 0 \Rightarrow \frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}$

Math 100A – WORKSHEET 8
CURVE SKETCHING

1. Convexity and Concavity

(1) Consider the curve $y = x^3 - x$.

(a) Find the line tangent to the curve at $x = 1$.

$\frac{dy}{dx} = 3x^2 - 1$ so $y'(1) = 2$, line is $y = 2(x-1)$

(positive, so graph increasing)

(b) Near $x = 1$, is the line above or below the curve? Hint: how does the slope of the curve behave to the right and left of the point?

$\frac{d^2y}{dx^2} = 6x$ so $y''(1) = 6 > 0$, so y' increasing around $x = 1$

so on right curve growing faster than line $\Rightarrow$ above it

on left curve " slower " " $\Rightarrow$ above it

General fact: if $f''(x) > 0$, tangent line is (locally) below graph

if $f''(x) < 0$, locally above

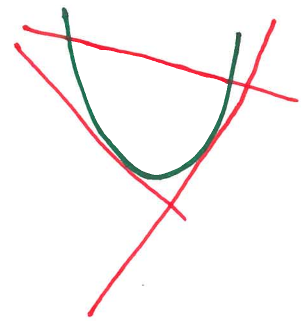
Date: 23/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Convexity

On an interval where $f''>0$ (more generally, f' increasing)

- tangent lines **below** graph
- secant lines **above** graph

Call f **convex** or **concave up**.

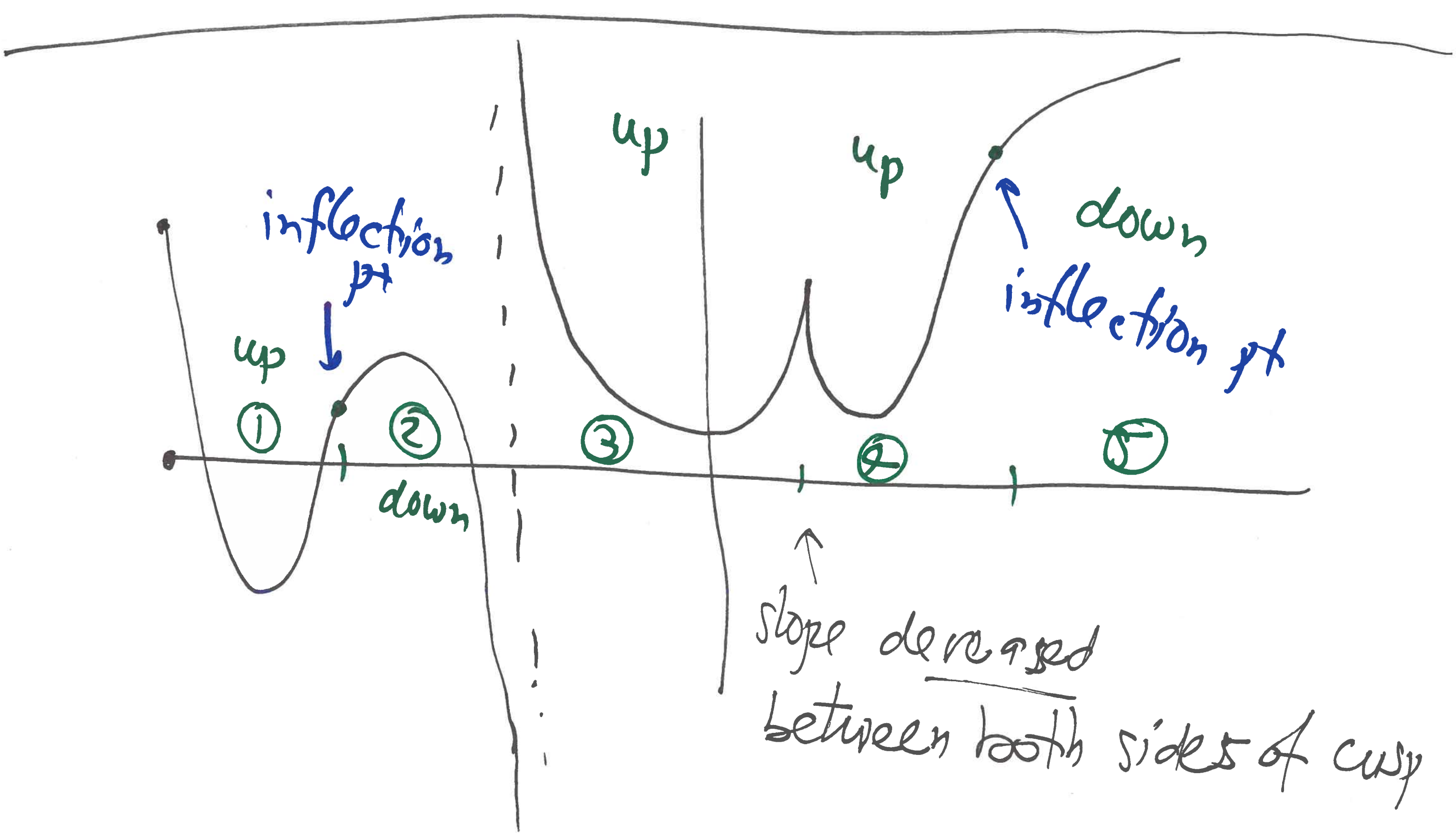


On an interval where $f''<0$ (more generally, f' is decreasing)

- tangent lines **above** graph
- secant lines **below** graph

Call f **concave** or **concave down**.

(flip picture)

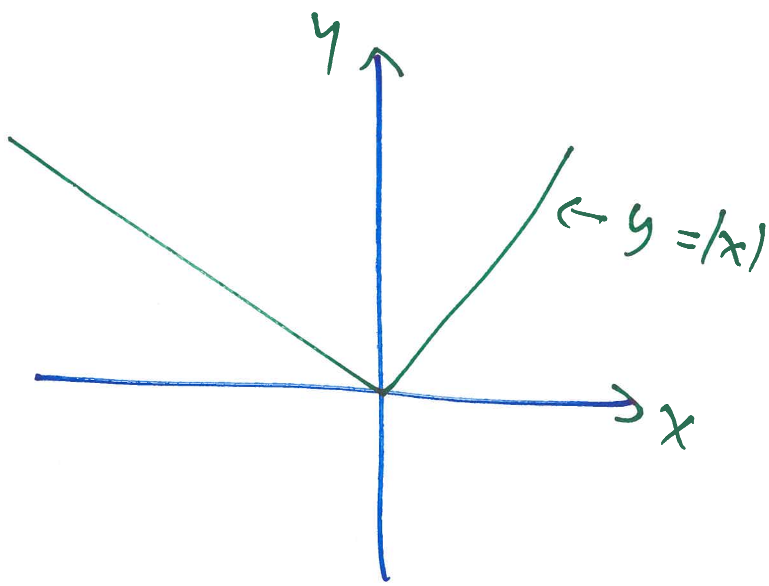


A point $(x_0, f(x_0))$ on curve is an inflection point if f is continuous there, and there is a change in concavity (up → down or down → up)

At an inflection point either $f''(x_0) = 0$
or $f''(x_0)$ DNE.

"potential inflection pt."

(**But**; $y = x^4$, $y''(0) = 0$ but no inflection pt)



← Concave up, $f''(0)$ DNE.

(2) For each curve find its domain; where is it concave up or down? Where are the inflection points.

(a) $y = x \log x - \frac{1}{2}x^2$.

Defined when $x > 0$ or on $(0, \infty)$ (because of log)

$y' = \log x + 1 - x$, $y'' = \frac{1}{x} - 1 = \frac{1}{x}(1-x)$

factor if you can

$\frac{1}{x}$ always positive here, so

on $(0,1)$ $y'' > 0$ concave up

on $(1, \infty)$ $y'' < 0$ concave down

The point $(1, -\frac{1}{2})$ is an inflection pt.

(b) $y = \sqrt[3]{x}$.

$y' = \frac{1}{3} x^{-2/3}$, $y'' = -2/9\, x^{-5/3}$

sign of $x^{-5/3}$ = sign of x

(y defined on $(-\infty, \infty)$)

so on $(-\infty, 0)$, $y'' > 0$ concave up

on $(0, \infty)$, $y'' < 0$ concave down

Inflection point at $(0,0)$ [$\sqrt[3]{x}$ is cts there]

---

or: $y''(-1) = 2/9 > 0$

$y''(1) = -2/9 < 0$

only "bad" point is $x = 0$

so $y'' > 0$ on $(-\infty, 0)$

< 0 on $(0, \infty)$

1 Curve Sketching notes

1.1 Tools. Let f be differentiable as needed on (a, b).

Fact (First derivative). (1) *If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly* increasing *there.*
(2) *If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly* decreasing *there.*

Every change involves either a *critical point* (f' vanishes) or a *singularity* (f' undefined).

Fact (Second derivative). (1) *If $f''(x) > 0$ for all $x \in (a, b)$ then f is* concave up *there.*
(2) *If $f''(x) < 0$ for all $x \in (a, b)$ then f is* concave down *there.*

Definition. A change in concavity is called an *inflection point.*

Theorem. *(Tests for minima and maxima) Let $x_0 \in (a, b)$ be a critical or singular number for f, and suppose f is continuous at x_0, differentiable near it.*

(1) *Either of the following is sufficient to show that f has a local* minimum *at x_0:*
 (a) $f''(x_0) > 0$ *or;*
 (b) *$f'(x)$ is negative to the left of x_0, positive to its right.*
(2) *Either of the following shows that f has a local* maximum *at x_0:*
 (a) $f''(x_0) < 0$ *or;*
 (b) *$f'(x)$ is positive to the left of x_0, negative to its right.*

1.2 Curve sketching protocol. Given a function f.

0th derivative stuff:
 (a) The domain and the domain of continuity.
 (b) Domains where $f > 0$, $f < 0$.
 (c) Anchor points: x- and y-intercepts.
 (d) vertical asymptotes.
 (e) Asymptotics at $\pm\infty$ (if in the domain)

1st derivative stuff:
 (a) Evaluate $f'(x)$ **[high stakes: error here loses a lot of points down the line]**
 Using this, determine:
 (b) Domains where $f' > 0$, $f' < 0$
 (c) Critical and singular points.

2nd derivative stuff:
 (a) Domains where $f'' > 0$, $f'' < 0$
 (b) Points where $f''(x) = 0$, inflection points.

Example: $y = x^4 - 4x^3$.

⓪ domain: $(-\infty, \infty)$ no vertical asymptotes

$y = x^3(x-4)$ cross axis at $x = 0, 4$.

On $(-\infty, 0)$ $y > 0$ $[x^3 < 0,\ x-4 < 0]$

On $(0, 4)$ $y < 0$ $[x^3 > 0,\ x-4 < 0]$

on $(4, \infty)$ $y > 0$ $[x^3 > 0,\ x-4 > 0]$

As $x \to \infty$, $y \sim x^4$, same as $x \to -\infty$.

① $y' = 4x^3 - 4 \cdot 3x^2 = 4x^2(x-3)$

critical pts at $(0,0)$, $(3,-27)$, no singular pts

since $x^2 \geq 0$, have $y' < 0$ on $(-\infty, 0)$, $(0, 3)$ decreasing

$y' > 0$ on $(3, \infty)$ increasing

② $y'' = 4 \cdot 3 \cdot x^2 - 4 \cdot 3 \cdot 2x = 12x(x-2)$

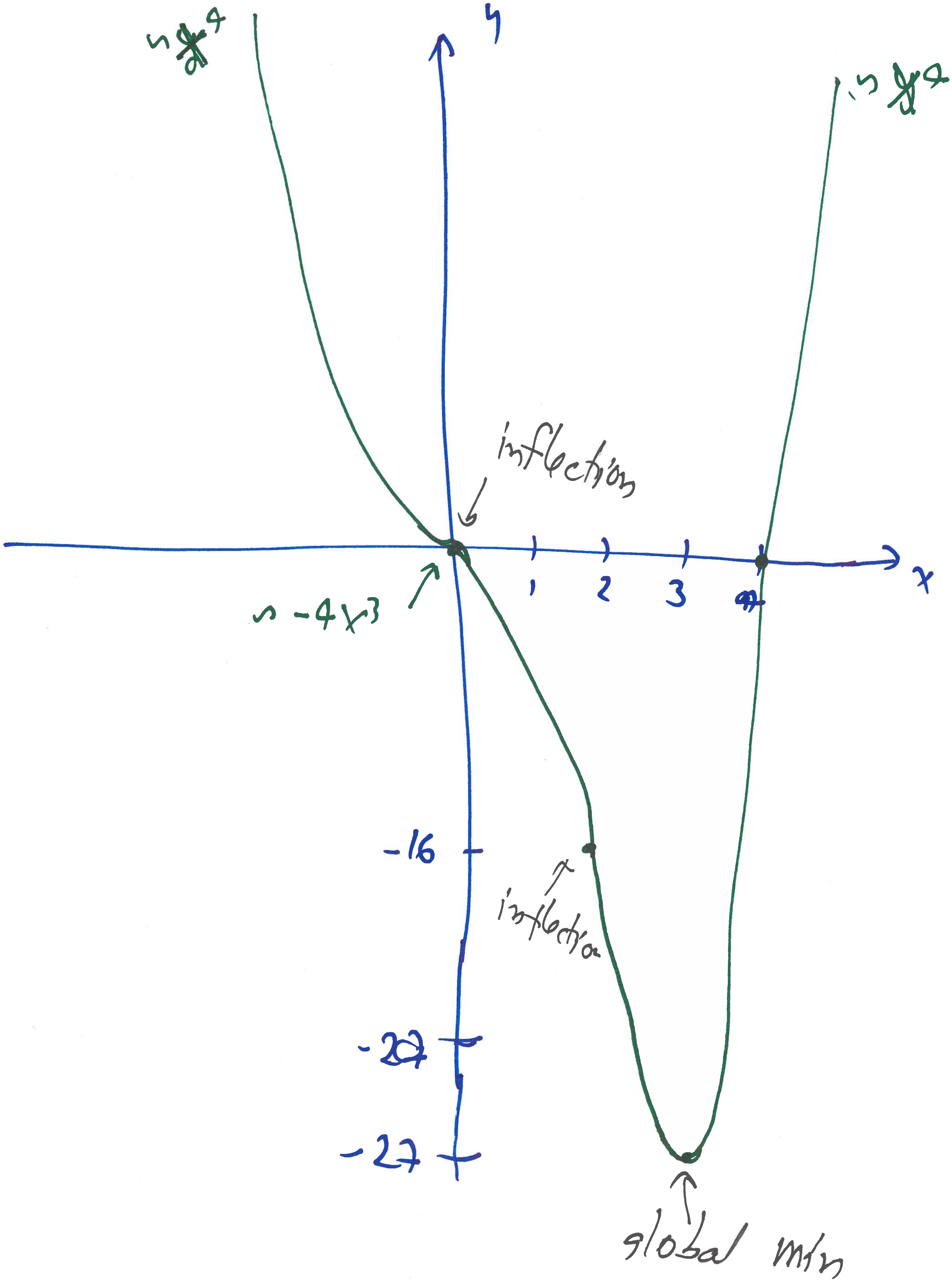
vanishes at $x = 0, 2$

on $(-\infty, 0)$ $y'' > 0$ concave up

On $(0, 2)$ $y'' < 0$ concave down

on $(2, \infty)$ $y'' > 0$ concave up

$\Rightarrow$ inflection pts at $(0,0)$, $(2, -16)$



y
inflection
x
1
2
3
4
~ -4x^3
-16
-27
-27
inflection
global min

2. Curve sketching

(3) Let $f(x) = \frac{x^2}{x^2+1}$ for which $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotics there?

Since $x^2+1 \geq 1 > 0$ for all x, f is defined everywhere
Since $x^2 > 0$ if $x \neq 0$, f only meets axes at $(0,0)$.
As $x \to \pm\infty$ $\frac{x^2}{x^2+1} \sim \frac{x^2}{x^2} \sim 1$ (horizontal asymptote!!)
no vertical asymptotes

(b) What are the intervals of increase/decrease? The local and global extrema?

Since $(x^2+1) > 0$, f' has same sign as x
so f is decreasing on $(-\infty, 0)$, increasing on $(0,\infty)$,
has its global minimum at $(0,0)$

(c) What are the intervals of concavity? Any inflection points?

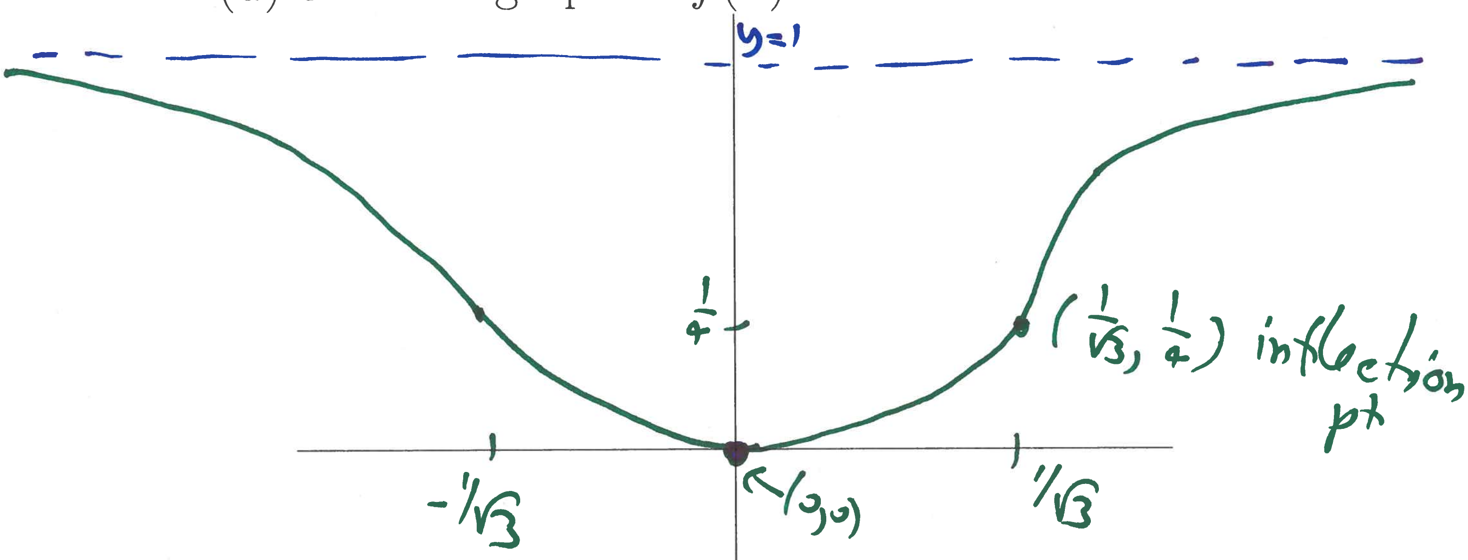
$f''(x) > 0 \iff 1-3x^2 > 0 \iff 3x^2 < 1 \iff |x| < \frac{1}{\sqrt{3}}$

so concave down on $(-\infty, -\frac{1}{\sqrt{3}})$, and on $(\frac{1}{\sqrt{3}}, \infty)$

concave up on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

Inflection points over $x = \pm\frac{1}{\sqrt{3}}$, i.e. at $(\pm\frac{1}{\sqrt{3}}, \frac{1}{4})$

(d) Sketch a graph of $f(x)$.



observe, here $y(-x) = y(x)$ say y is an even function of x.

(odd : $f(-x) = -f(x)$)

(4) Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotics there?

Defined, cts on $(-\infty, \infty)$, always > 0 since $e^u > 0$ for all u.

As $x \to \pm\infty$, $-\frac{(x-\mu)^2}{2\sigma^2} \sim -\frac{x^2}{2\sigma^2} \xrightarrow[x\to\pm\infty]{} -\infty$

so $e^{-\frac{(x-\mu)^2}{2\sigma^2}} \xrightarrow[x\to\pm\infty]{} 0$ [but no simpler asymptotics]

no vertical asymptotes

(b) What are the intervals of increase/decrease? The local and global extrema?

$$f'(x) = \frac{-1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)$$

so on $(-\infty, \mu)$ $f'(x) > 0$, f increasing

on (μ, ∞) $f'(x) < 0$, f decreasing

critical point (global max) at $\left(\mu, \frac{1}{\sqrt{2\pi\sigma^2}}\right)$

$$f''(x) = \frac{1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left(\left(\frac{x-\mu}{\sigma}\right)^2 - 1\right)$$

(c) What are the intervals of concavity? Any inflection points?

f'' cts everywhere, vanishes where $\left(\frac{x-\mu}{\sigma}\right)^2 = 1$, $(x-\mu)^2 = \sigma^2$

so at $x = \mu \pm \sigma$ (because $e^u > 0$, $\frac{1}{\sqrt{2\pi\sigma^6}} > 0$,

As $x \to \pm\infty$, $\left(\frac{x-\mu}{\sigma}\right)^2 - 1 \sim \frac{x^2}{\sigma^2} > 0$ so

$(-\infty, \mu-\sigma)$: $f'' > 0$, concave up

$(\mu-\sigma, \mu+\sigma)$: $f''(\mu) = -\frac{1}{\sqrt{2\pi\sigma^6}} < 0$ so $f'' < 0$, concave down

$(\mu+\sigma, \infty)$: $f'' > 0$, concave up

(d) Sketch a graph of $f(x)$.