9. Optimization (1/11/2024)

Goals.

(1) Review: calculus and the shape of the graph
(2) Optimization of functions
(3) Problem solving: optimization problems

Last Time. Curve Sketching

If f is diff on $[a,b]$: increasing if $f'>0$, decreasing if $f'<0$ $\Rightarrow$ if x_0 is a local max/min then x_0 is one of:

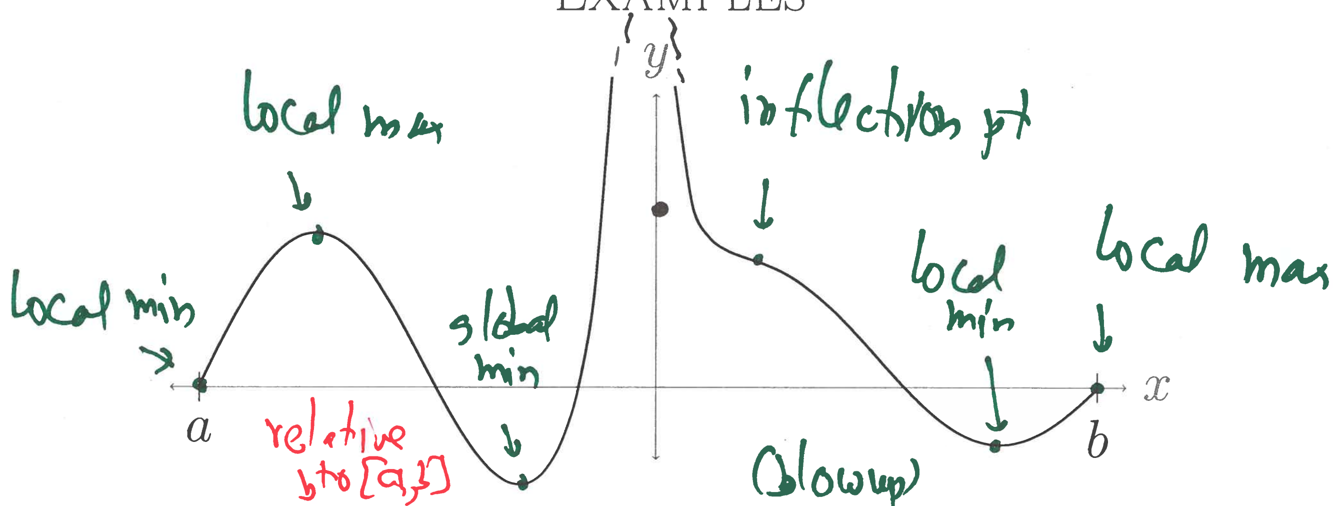
$f'(x_0)=0$	$f'(x_0)$ DNE	$x_0=a$, $x_0=b$
Critical pt	singular pt	endpoint

(point: $(x_0, f(x_0))$)

Also can tell if local max/min ① via f' (increase/decrease on both sides)

② via $f''(x_0)<0 \Rightarrow$ local max

EXAMPLES



(1) Is there a *global* max on $[a, b]$? no Is there a global min? yes

(2) How many *local* maxima are there? 2 local minima? 3

relative to nearby pts

Which point on the line $3x + y = 6$ is closest to the point $(7, 5)$?

calculus: where is $f(x) = (x-7)^2 + (6-3x-5)^2$ smallest?

smallest dist → smallest $(\text{dist})^2$

parametrize point using x-coord then $y = 6-3x$

f diff everywhere ⇒ min at endpoints or critical pts: as $x \to \pm\infty$ $f(x) \sim 10x^2 \to \infty$

l'Hôpital's rule

Statement: ① Say $\lim_{x\to a} f(x), g(x)$ both 0 or both $\pm\infty$

② suppose $f'(x), g'(x)$ exist near a.

③ " $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ (exists and $= L$)

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$

(also works if a is ∞, if L "$=$" $\pm\infty$.)

Example continued

$f'(x) = 2(x-7) + 2\cdot 3(3x-1) = 20x - 20 = 20(x-1)$

only crit pt at $x=1$, so min at $(1,3)$

Fact ("closed interval method")

Let f be continuous on $[a,b]$

Then ① f has a global max & a global min. on $[a,b]$.

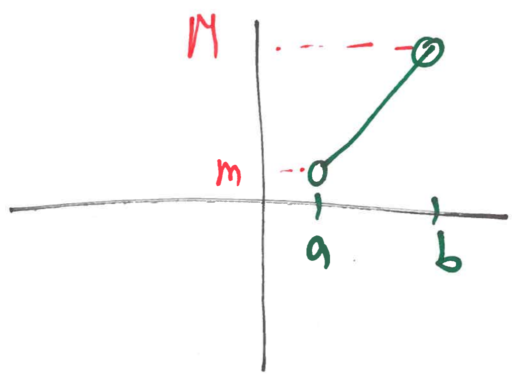
② These occur at one of
critical pt
singular pt
endpoint

③ global max value = largest value seen at those points

(same for min)

If interval not closed, need to handle endpoints, often by asymptotics.
Need not have max or min

E.g.



Math 100A – WORKSHEET 9
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

f is a polynomial $\Rightarrow$ everywhere diff.

$f'(x) = 4x^3 - 8x = 4x(x^2-2)$, vanishes at $0, \pm\sqrt{2}$

$f(\pm 5) = 625 - 100 + 4 = 529$ ← absolute max on $[-5,5]$

$f(\pm\sqrt{2}) = 0$ ← absolute min on $[-5,5]$

$f(0) = 4$

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

critical pts over $x=0$

$f(\pm 1) = 1$ ← absolute min on $[-1,1]$

$f(0) = 4$ ← absolute max on $[-1,1]$

Date: 30/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Common error: not checking if a zero of f' is in the domain

$f' = (1+x)\sqrt{x}$ does not vanish at $x=-1$

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

max is still 4 ($f(0) = 4$ is maximal on $[-1,1]$,

no min (on $[-1,1]$ min value was only attained at endpoints)

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

3 critical pts $0, \pm\sqrt{2}$

$f(0) = 4$, $f(\pm\sqrt{2}) = 0$, as $x \to \pm\infty$ $f(x) \sim x^4 \to \infty$

no max, min $= 0$.

Optimization / Related Rates notes

(0) Read problem: understand the idea, draw a picture if possible.
(1) Assign names:
   - Choose axes, quantities of interest.
   - Give a *name* to each quantity of interest. , domain
(2) Function/relations: express quantity to be optimized as a function of the dependent variable.
   - Sometimes the quantity depends on several variables, and we need to enforce *relations* between them to end up with one independent variable.
(3) Calculus: find the (relevant) domain of the objective function and the minima and maxima on the domain.
   - (Related rates: use the chain rule when differentiating).
(4) Interpretation: solve the problem using the calculus result.
   - Make *sanity checks* (area can't be negative, for example).

(4) A fish swimming at speed v relative to the water faces a drag force of the form av^2 and thus has to output a power of av^3. If the fish is swimming against a current of speed $u > 0$ (thus with speed $v > u$), it will cover a distance L at time $\frac{L}{v-u}$. The total energy cost is then $E = av^3 \frac{L}{v-u}$. At what speed v should the fish swim to minimize this cost?

Need min $E(v) = aL \frac{v^3}{v-u}$ on (u, ∞)

As $v \to u$, $E(v) \sim aLu^3 \frac{1}{v-u} \to \infty$

As $v \to \infty$, $E(v) \sim aL\, v^2 \to \infty$

so must have min in interior (E(v) is continuous)

$$E'(v) = aL \frac{3v^2(v-u) - v^3}{(v-u)^2} = \frac{aLv^2}{(v-u)^2}(2v-3u)$$

vanishes iff $2v - 3u = 0$, i.e. iff $v = \frac{3}{2}u$

Only one crit pt so min at $\boxed{\frac{3}{2}u.}$

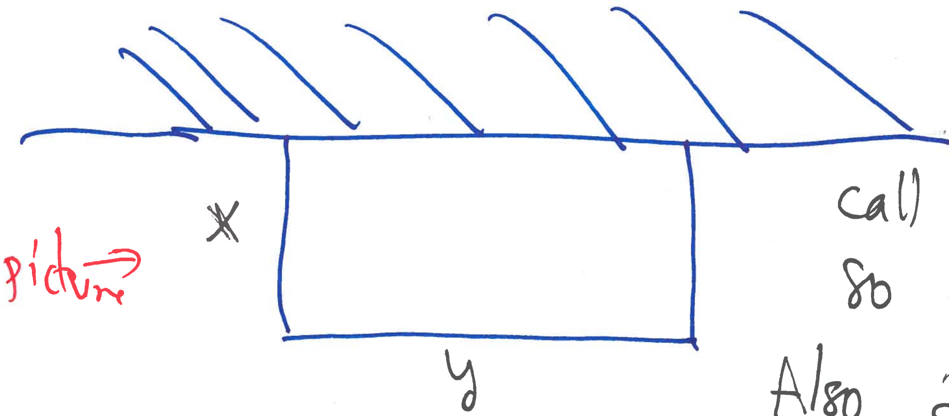
(5) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

(a) What is the range of r values that makes sense?

(6) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?

names

picture



call the sides x, y

so area is $A = xy$

Also $2x + y = 100$

relations

Then $y = 100 - 2x$ so $A = x(100-2x)$

want the max of A if $0 \le x \le 50$, ie on $[0, 50]$

objective function

domain

Natural to admit degenerate rectangles $(0 \times 100,\ 50 \times 0)$ to get closed interval

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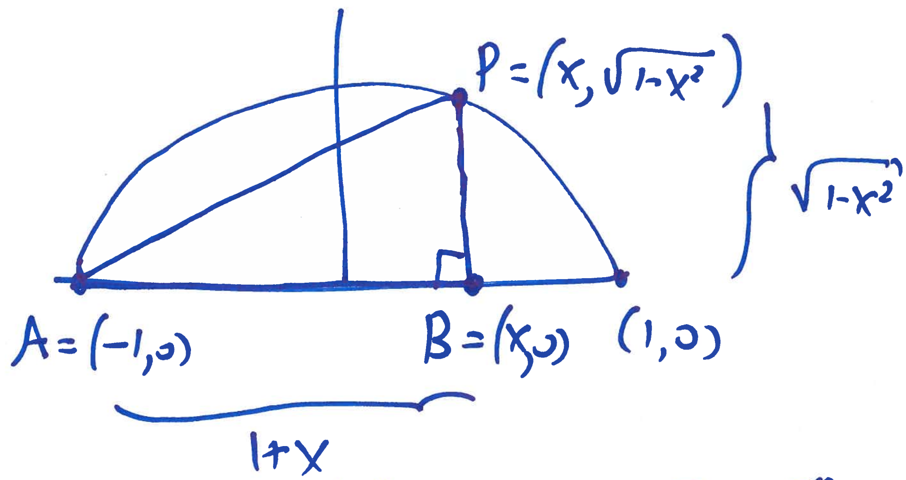
Now $A = 100x - 2x^2 = -2(x^2 - 50x) = -2(x^2 - 50x + 25)^2 + 2 \cdot 25^2$

$= 1{,}250 - 2(x-25)^2 \le 1{,}250$

So max area is $1{,}250\,m^2$ attained at $x = 25m$

Or: $A'(x) = 100 - 4x$, crit pt at 25, $A(25) = 1{,}250\,m^2$, $A(0) = A(50) = 0$

(8) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1-x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?



parametrize triangles by x,

$-1 \le x \le 1$

So area is $f(x) = \frac{1}{2}(1+x)\sqrt{1-x^2}$ want max on $[-1,1]$

$$f'(x) = \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}(1+x)\frac{2x}{2\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}}\left(1-x^2-x-x^2\right)$$

$$= \frac{1-x-2x^2}{2\sqrt{1-x^2}}$$

crit pts where $1-x-2x^2=0$

i.e. $\frac{1 \pm \sqrt{1+8}}{-4} = \frac{1 \pm 3}{-4} = -1, \frac{1}{2}$

$f(-1) = 0, \quad f(1) = 0$

$f\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{3\sqrt{3}}{8}.$

So largest area is $\frac{3\sqrt{3}}{8}$.