10. Taylor expansion (4/11/2024)

Goals.

(1) Review: Linear approximation
(2) Higher order approximation
(3) Manipulating expansions

Last Time. Optimization

Linear approximation: we can approximate f near a by $f(x) \simeq f(a) + f'(a)(x-a)$

earlier lecture:
1. from this get $f'(a)$
2. given $f'(a)$ find approximate $f(x)$

Today: improve this with higher-order terms

(worksheet available at https://personal.math.ubc.ca/~lior/teaching/2425/100_F24/#schedule)

1. Taylor expansion

(1) (Review) Use linear approximations to estimate:

(a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

Think of $f(x) = \log x$, want $f(\frac{2}{3}), f(\frac{4}{3})$

notice $\frac{2}{3}, \frac{4}{3}$ close to $a=1$

Have $f'(x) = \frac{1}{x}$, so $f(1) = \log 1 = 0$, $f'(1) = \frac{1}{1} = 1$,

for x near 1, $f(x) \approx 0 + 1\cdot(x-1) = (x-1)$,

so $f(\frac{2}{3}) \approx (\frac{2}{3}-1) = -\frac{1}{3}$, $f(\frac{4}{3}) \approx (\frac{4}{3}-1) = \frac{1}{3}$

so $\log 2 = \log\frac{4/3}{2/3} = \log\frac{4}{3} - \log\frac{2}{3} \approx 2/3$

(b) $\sin 0.1$ and $\cos 0.1$.

$(\sin\theta)' = \cos\theta$, $(\cos\theta)' = -\sin\theta$

At $a=0$, $\sin 0 = 0$, $\cos 0 = 1$ so $\sin\theta \approx \theta$

$\cos 0 = 1$, $-\sin 0 = 0$, so $\cos\theta \approx 1 + 0\cdot\theta \approx 1$

so to 1^{st} order, $\sin 0.1 \approx 0.1$, $\cos 0.1 \approx 1$.

linear approx can be constants

Discussion, further example.

① What does it mean that

$f(x) \doteq f(a) + f'(a)(x-a)$?

⓪ $f(x) \approx f(a)$ as $x \to a$

encondes continuity

① linear term expresses asymptotics of $f(x) - f(a)$ as $x \to a$

$\Rightarrow$ "Error" $f(x) - (f(a) + f'(a)(x-a))$

decay faster that $x-a$ as $x \to a$

Problem: find $\lim_{x \to \infty} ax \tan\left(\frac{b}{x}\right)$ $(a, b \neq 0)$

Solution: as $x \to \infty$ $ax \to \infty$, $\frac{b}{x} \to 0$ "small param"

so $\tan\left(\frac{b}{x}\right) \to 0$

At $a=0$, $[\tan\theta]_{\theta=0} = 0$, $[(\tan\theta)']_{\theta=0} = \left[\frac{1}{\cos^2\theta}\right]_{\theta=0} = 1$

so $\tan\theta \doteq \theta$ as $\theta \to 0$, so $\tan\left(\frac{b}{x}\right) \doteq \frac{b}{x}$ as $x \to \infty$

$\Rightarrow ax \tan\left(\frac{b}{x}\right) \sim (ax)\cdot\left(\frac{b}{x}\right) \sim ab.$

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(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(a) $f'(x) = e^x$, $f^{(2)}(x) = e^x$, $f^{(3)}(x) = e^x$, ..

so $f(0) = f'(0) = f^{(2)}(0) = f^{(3)}(0) = \dots = 1$

(b) $f(x) \not\equiv 1$; Take $T_0(x) \equiv 1$

then $T_0(0) = 1 = f(0)$

(c) try $T_1(x) = a + bx$

Then $T_1(0) = a$ want $T_1(0) = 1$ so take $a=1$

$T_1'(0) = b$ want $T_1'(0) = 1$, so take $b=1$.

Get $T_1(x) = 1 + x$

(d) Try $T_2(x) = a + bx + cx^2$.

Need $a = T_2(0) = f(0) = 1$, $T_2'(x) = b + 2cx$

so need $b = T_2'(0) = f'(0) = 1$

Also $T_2''(x) = 2c$ so need $c = \frac{1}{2}$ to get $T_2''(0) = 1$

(c) Try $T_3(x) = 1 + x + \frac{1}{2}x^2 + dx^3$

Then $T_3(0) = 1$ ✓

$T_3'(x) = 1 + x + 3dx^2$; $T_3'(0) = 1$ ✓

$T_3''(x) = 1 + 6dx$; $T_3''(0) = 1$ ✓

$T_3^{(3)}(x) = 6d$; to get $T_3^{(3)}(0) = 1$

need $\boxed{d = \frac{1}{6}}$

$$T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$6 = 1 \cdot 2 \cdot 3 = 3!$$

<u>Remark here:</u> if we keep going, the coeff of x^k will be $\frac{1}{1 \cdot 2 \cdot 3 \cdots k}$

call $1 \cdot 2 \cdot 3 \cdots k$ "k factorial"

write $k!$.

(3) Do the same with $f(x) = \log x$ about $x = 1$.

To 3^{rd} order at $x=1$,

$$\log x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

Examine calc:

we want $T_n(a) = f(a)$

so that's constant term

$\vdots$

see: k'th term is

$$\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Key fact: $f(x) \approx T_n(x)$ in sense that as $x \to a$, $f(x) - T_n(x) \ll (x-a)^n$

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) ⋆ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$; $f(0) = 1$

$f'(x) = 1 (1-x)^{-2}$; $f'(0) = 1$

$f^{(2)}(x) = 1 \cdot 2 \, (1-x)^{-3}$; $f''(0) = 1 \cdot 2$

$f^{(3)}(x) = 1 \cdot 2 \cdot 3 \, (1-x)^{-4}$; $f^{(3)}(0) = 1 \cdot 2 \cdot 3$

$f^{(4)}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \, (1-x)^{-4}$; $f^{(4)}(0) = 1 \cdot 2 \cdot 3 \cdot 4$

So

$$T_4(x) = 1 + 1 \cdot x + \frac{1 \cdot 2}{2} \cdot x^2 + \frac{1 \cdot 2 \cdot 3}{3!} x^3 + \frac{1 \cdot 2 \cdot 3 \cdot 4}{4!} \cdot x^4$$

$$= 1 + x + x^2 + x^3 + x^4.$$

MacLaurin expansion = Taylor exp. about $x = 0$

(5) Find the nth order MacLaurin expansion of $\cos x$, and approximate $\cos 0.1$ using the 3rd order expansion

Let $f(x) = \cos x$. Then $f'(x) = -\sin x$

$f^{(2)}(x) = -\cos x$, $f^{(3)}(x) = \sin x$

$f^{(4)}(x) = \cos x$ so repeat from now on

$\Rightarrow f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f^{(3)}(0) = 0,$ repeat.

$$\Rightarrow 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

To 3^{rd} order, $\cos x \approx 1 - \frac{1}{2}x^2$

$$\cos 0.1 \approx 1 - \frac{1}{2}\cdot(0.1)^2 = \frac{199}{200}.$$

Facts

n'th order expansion is

$$f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Common errors: ① only writing n'th order term.

② ("non linear line") using $f^{(k)}(x)$ where we want $f^{(k)}(a)$

not true: $\log x \stackrel{?}{=} 0 + \frac{1}{x}(x-1)$

log 1 correct

$(\log x)'$ instead of value at $a=1$

Sanity check:

is what we wrote a polynomial of degree n?

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \ldots$$

$$\log x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$$

$$\log(1+u) \approx u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \ldots$$

$$\frac{1}{1-u} \approx 1 + u + u^2 + u^3 + u^4 + \ldots$$

$$\cos\phi \approx 1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 + \ldots$$

$$\sin\phi \approx \phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \frac{1}{7!}\phi^7 + \ldots$$

Memorize!

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Since $\frac{f''(3)}{2!} = 12$, $f''(3) = 24$.

Common error $f''(3) \neq 12$

(7) In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the "rest mass" of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

$E(x) = mc^2(1-x)^{-\frac{1}{2}}$, $x = v^2/c^2$

$E'(x) = \frac{1}{2}mc^2(1-x)^{-3/2}$; $E''(x) = \frac{3}{4}mc^2(1-x)^{-5/2}$

So $E(0) = mc^2$, $E'(0) = \frac{1}{2}mc^2$, $E''(0) = \frac{3}{4}mc^2$

So to 2^{nd} order in x $E(x) \approx mc^2 + \frac{1}{2}mc^2 \cdot x + \frac{3}{8}mc^2 x^2$

to 4^{th} order in v

So $E(v) \approx mc^2 + \frac{1}{2}mc^2\frac{v^2}{c^2} + \frac{3}{8}mc^2\frac{v^4}{c^4}$

$= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2 \cdot \frac{v^2}{c^2}$

2. New expansions from old

Near $u = 0$: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$
$\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$
$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \cdots$
$\sin u = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7 + \cdots$ $\cos u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \frac{1}{6!}u^6 + \cdots$

(8) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

To 3rd order $\sin\theta \approx \theta - \frac{1}{6}\theta^3$

so $\sin\frac{1}{100} \approx \frac{1}{100} - \frac{1}{6\cdot 10^6}$.

Know $x - \frac{1}{6}x^3$ is close to $\sin x$ is x small
(3rd-order close)

$1+x$ is close to $1+x$

so $(1+x)(x - \frac{1}{6}x^3)$ is close to $(1+x)\sin x$.

To 3rd order in x,

$(x+1)\sin x \approx (1+x)(x - \frac{1}{6}x^3) \approx x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4$

$\approx \boxed{x + x^2 - \frac{1}{6}x^3}$

Can multiply & add expansions
(work to desired order)

(9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

Recalls $e^u \approx 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$

$\frac{1}{1-v} = 1 + v + v^2 + v^3 + v^4 + \cdots$

So to 8th order in x used $u = x^2$

$e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8$

$\frac{1}{1+x^3} \approx 1 + (-x)^3 + (-x^3)^2 \approx 1 - x^3 + x^6$

used $v = -x^3$

So $e^{x^2} - \frac{1}{1+x^3} \approx x^2 + x^3 + \frac{1}{2}x^4 - \frac{5}{6}x^6 + \frac{1}{24}x^8$

So $\frac{f^{(6)}(0)}{6!} = -5/6$ so $f^{(6)}(0) = -5 \cdot \frac{6!}{6} = -600$

(11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about $x = 0$.

idea: as $x \to 0$, $\cos(3x) \to 1$

so this is $\frac{1}{1+(\cos 3x - 1)} = \frac{1}{1-(1-\cos 3x)}$

① approx $\cos 3x$ to 4th order (using approx for $\cos\phi$)

② find v to use in $\frac{1}{1-v}$ ...

(12) (Change of variable/rebasing polynomials)

(a) Find the Taylor expansion of the polynomial $x^3 - x$ about $a = 1$ using the identity $x = 1 + (x-1)$.

$$x^3 - x = (1+(x-1))^3 - (1+(x-1))$$
$$= 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3 - 1 - (x-1)$$
$$= 2(x-1) + 3(x-1)^2 + (x-1)^3$$

(b) Expand e^{x^3-x} to third order about $a = 1$.

Know $e^u \approx 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3$ to 3^{rd} order,

use $u = 2(x-1) + 3(x-1)^2 + (x-1)^3$

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$

(a) $\sqrt{x^4 + 3x^3} - x^2$

Notice: $x^4 + 3x^3 \sim x^4$ as $x \to \infty$

so $\sqrt{x^4 + 3x^3} \sim \sqrt{x^4} \sim x^2$

~~wrong~~: $\sqrt{x^4+3x^3} - x^2 \sim x^2 - x^2 = 0$

notice catastrophic cancellation

$\sqrt{x^4+3x^3} - x^2 = x^2\left(\sqrt{1+\frac{3}{x}} - 1\right)$

extract asymptotics

$3/x \to 0$ small parameter

~~(b) $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$~~

to 1st order, $\sqrt{1+u} \simeq 1 + \frac{1}{2}u$

so $\sqrt{1+u} - 1 \sim \frac{1}{2}u$ as $u \to 0$

so $\sqrt{1+\frac{3}{x}} - 1 \sim \frac{1}{2}\frac{3}{x}$ as $x \to \infty$

so $\sqrt{x^4+3x^3} - x^2 \sim x^2 \cdot \frac{3}{2x} \sim \frac{3}{2}x$

as $x \to \infty$

What about

$$\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$$

$$= x^2\left(\sqrt[3]{1-\frac{1}{x^2}} - \sqrt{1-\frac{2}{3}\frac{1}{x^2}}\right)$$

to 1st order in u

$$(1+u)^{\frac{1}{2}} \approx 1+\frac{1}{2}u, \quad (1+u)^{1/3} \approx 1+\frac{1}{3}u$$

So $\left(1-\frac{1}{x^2}\right)^{1/3} \approx 1 - \frac{1}{3}\frac{1}{x^2}$

$$\left(1-\frac{2}{3}\frac{1}{x^2}\right)^{1/2} \approx 1 - \frac{1}{3}\frac{1}{x^2}$$

to 2nd order, $(1+u)^{\frac{1}{2}} \approx 1+\frac{1}{2}u - \frac{1}{8}u^2$

$$(1+u)^{1/3} \approx 1+\frac{1}{3}u - \frac{1}{9}u^2$$

So $\left(1-\frac{1}{x^2}\right)^{1/3} \approx 1 - \frac{1}{3}\frac{1}{x^2} - \frac{1}{9}\frac{1}{x^4}$

$$\left(1-\frac{2}{3}\frac{1}{x^2}\right)^{1/2} \approx 1 - \frac{1}{3}\frac{1}{x^2} - \frac{1}{18}\frac{1}{x^4}$$

So $\sqrt[3]{\ } - \sqrt[2]{\ } \sim x^2(\text{diff}) \sim x^2\left(-\frac{1}{18x^4}\right) \sim -\frac{1}{18x^2}$.

(16) Evaluate $\lim_{x\to 0} \frac{e^{-x^2/2} - \cos x}{x^4}$.

To 4th order, $e^u \approx 1 + u + \frac{1}{2}u^2$

$$e^{(-x^2/2)} \approx 1 - \frac{x^2}{2} + \frac{1}{8}x^4$$

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{1}{24}x^4$$

So $e^{-x^2/2} - \cos x \approx \frac{1}{12}x^4$ to 4th order

So $\frac{e^{-x^2/2} - \cos x}{x^4} \approx \frac{1}{12}$ to 0th order

So $\lim_{x\to 0} \frac{e^{-x^2/2} - \cos x}{x^4} = \frac{1}{12}$