MATH 100 - Section 1A3, lecture 12

20/11/2024

No more midterms/ group projects

Newton's Method

(Also SC12, see chapter 10 of textbook)

Problem: We want to solve equations, but we can't.

Examples: $x^2 = 3 \iff x^2 - 3 = 0$

$2x - \sin x = y$ say $2x - \sin x = \pi - 1$

want $e^x = \sin x$

① bisection

1st example study $f(x) = x^2 - 3$, want to solve $f(x) = 0$

[solve $f(x) - g(x) = 0$ rather than $f(x) = g(x)$]

Observe: $f(1) = 1^2 - 3 = -2 < 0$

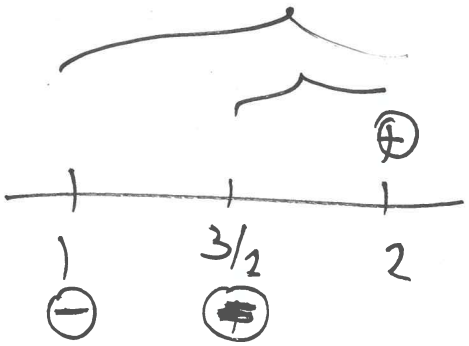
$f(2) = 2^2 - 3 = 1 > 0$

by continuity, f must cross axis between $[1, 2]$

(bracketed solution by 1, 2)

Idea: try midpoint $\frac{3}{2}$.

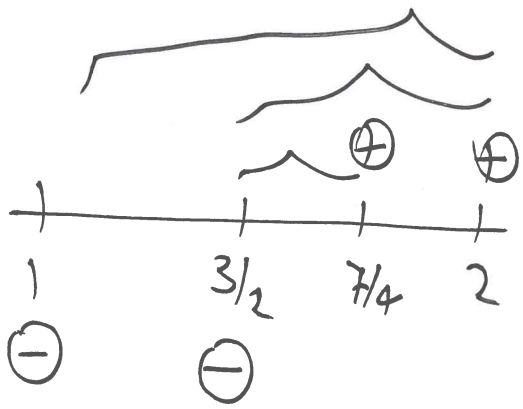
$$f\left(\frac{3}{2}\right) = \frac{9}{4} - 3 = -\frac{3}{4} < 0$$



$\Rightarrow$ must be root between $3/2$ & 2

Next: try $\frac{2+3/2}{2} = 7/4$

$$f\left(\frac{7}{4}\right) = \frac{49}{16} - 3 = \frac{1}{16} > 0$$



Observation: This is an iterative method

② Newton's method

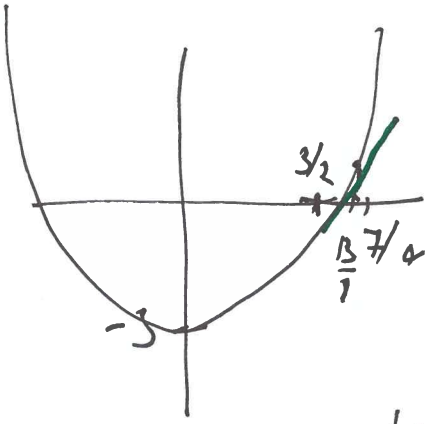
Start with guess $\frac{13}{8}$

linear approx to $f(x) = x^2 - 3$ at $\frac{13}{8}$:

$f'(x) = 2x$

$f(\frac{13}{8}) = -\frac{23}{64}$

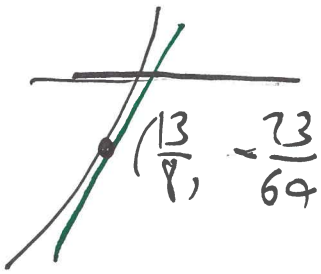
$f'(\frac{13}{8}) = \frac{13}{4}$



so $f(x) \approx -\frac{23}{64} + \frac{13}{4}(x - \frac{13}{8})$

find solution to $\frac{13}{4}(x - \frac{13}{8}) = \frac{23}{64}$

so $x = \frac{13}{8} + \frac{23/64}{13/4}$



Now iterate.

In general

Have guess x_n linear approx is:

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n)$$

So $f(x) \approx 0$ if $f(x_n) + f'(x_n)(x - x_n) = 0$

$$\Rightarrow \text{if } x = x_n - \frac{f(x_n)}{f'(x_n)}$$

Method:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example. (Greek math) If $f(x) = x^2 - b$ set

$$x_{n+1} = \frac{1}{2}(x_n + b/x_n)$$