13. Differential Equations (27/11/2024)

Goals.

(1) Differential equations
   (a) What is a differential equation?
   (b) What is a solution to a differential equation?
   (c) Plugging ansatze into equations
(2) First order Linear DE

Last Time. Newton's Method

Problem: Want to solve $f(x)=0$

Idea: Improve guess x_n to $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

(also mentioned bisection)

based on approximating f by line through $(x_n, f(x$

Newton

Goal of mechanics: Want to find the behavior $y(t)$ of a physical system as a function of time.

[unknown is a function]

Laws of physics are equations for this function:

e.g. $my'' = F(y)$. [equation involves derivatives]

Math 100A – WORKSHEET 13
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

$3^2 = 9 \neq 4$ ×

$2^2 = 4$ ✓

$3^2 = 3 \cdot 3$ ✓

$2^2 = 4 \neq 6 = 3 \cdot 2$ ×

No need to solve equation to check if a guess works!

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

$\frac{d(x^2)}{dx} = 2x \neq x^2$ ×

$\frac{d(e^x)}{dx} = e^x$ ✓

$\left(\frac{d(x^2)}{dx}\right)^2 = (2x)^2 = 4(x^2)$ ✓

$\left(\frac{d(e^x)}{dx}\right)^2 = (e^x)^2 = e^{2x} \neq 4e^x$ ×

Date: 27/11/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(4) Which of the following (if any) is a solution of $\frac{dy}{dx} = \frac{x}{y}$

A. $y = -x$; B. $y = x+5$ C. $y = \sqrt{x^2+5}$

$\frac{d(-x)}{dx} = -1 = \frac{x}{-x}$ ✓

$\frac{d(x+5)}{dx} = 1 \neq \frac{x}{x+5}$ ✗

$\frac{d((x^2+5)^{1/2})}{dx} = \frac{1}{2}(x^2+5)^{-\frac{1}{2}} \cdot 2x$

$= \frac{x}{\sqrt{x^2+5}}$ ✓

Can have multiple solutions!

The exponential

Important equation: $\boxed{y' = ry}$ (r constant)

$$\frac{d(e^t)}{dt} = e^t \quad \text{so} \quad \frac{d(e^{rt})}{dt} = re^{rt}$$

In fact, $y(t) = Ce^{rt}$ works for any constant C!

General solution: $\boxed{y(t) = Ce^{rt}}$

If $r > 0$, solutions escape to $\pm\infty$

If $r < 0$, solution decay to 0

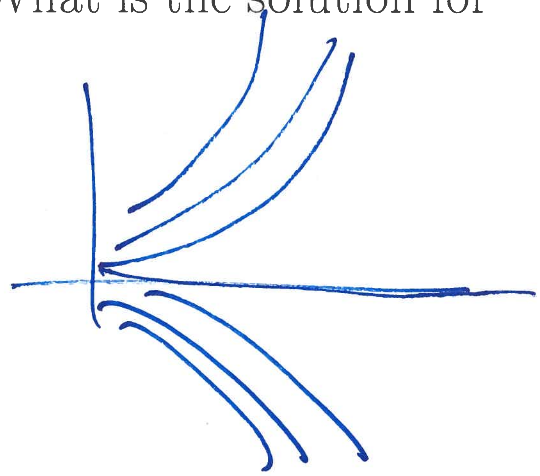
Given an initial condition say $y(0) = y_0$

Can find C: this means $Ce^{r \cdot 0} = y_0$ so $C = y_0$

get particular solution $\boxed{y(t) = y_0 e^{rt}}$

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

Solutions are: $Ce^{1.04t}$



want $y(0)=100$ need

$y(t) = 100e^{1.04t}$

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

Ansatz

2. Solutions by massaging and ansätze

(7) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

$$\frac{dy}{dt} = \omega \cdot \cos(\omega t), \quad \frac{d^2y}{dt^2} = -\omega^2 \sin(\omega t)$$

so $$\frac{d^2y}{dt^2} + 4y = (4 - \omega^2)\sin(\omega t) \leftarrow \text{plug into eqn.}$$

So $\omega = 2$ gives the solution $\boxed{\sin(2t).}$ ← solve for ω

In fact, $A\sin(2t)$ works for any A.

General solution: is $A\sin(2t) + B\cos(2t)$

$$(A\sin(2t))'' = -4A\sin(2t)$$

or $\frac{dy}{dt} = ay + d$

(9) Consider the equation $\frac{dy}{dt} = a(y-b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by u?

$$\frac{du}{dt} = \frac{dy}{dt} + 0 = a(y-b) = au$$

The DE is an indirect description of y.
Can still use it to argue about a solution y.

(b) What is the general solution for $u(t)$?

$$u(t) = Ce^{at}$$

(c) What is the general solution for $y(t)$?

$$y(t) = u(t) + b = b + Ce^{at}$$

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?

$y(t) \xrightarrow[t \to \infty]{} b$

("stable equillibrium/ steady state/ fixed point")

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?

$y(0) = b + C$ so $C = y(0) - b$

so $y(t) = b + (y(0) - b)e^{at}$

(10) Example: *Newton's law of cooling.* Suppose we place an object of temperature $T(0)$ in an environment of temperature T_{env}. It turns out that a good model for the temperature $T(t)$ of the object at time t is

$$\frac{dT}{dt} = -k\,(T - T_{\text{env}})$$

where $k > 0$ is a positive constant.

(a) Suppose $T(t) > T_{\text{env}}$. Is $T'(t)$ positive or negative? What if $T(t) < T_{\text{env}}$? Explain this in words.

if $T(t) > T_{env}$, $T - T_{env} > 0$ so $-k(T - T_{env}) < 0$

so $T'(t) < 0$

(b) A body is found at 1:30am and its temperature is measured to be 32.5°C. At 2:30am its temperature is found to be 30.3°C. The temperature of the room in which the body was found is measured to be 20°C and we have no reason to believe the ambient temperature has changed. What was the time of death?

Advice: work with variable $u(t) = T(t) - T_{env}$

facts $u(t) = Ce^{-kt}$

Advice; we choose when $t=0$ is

here choose $t=0$ at 1:30 am